

# Jordan decomposition and dynamics on flag manifolds

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## Abstract

Let  $\mathfrak{g}$  be a semisimple Lie algebra and  $G = \text{Int}(\mathfrak{g})$ . In this article, we relate the Jordan decomposition of  $X \in \mathfrak{g}$  (or  $g \in G$ ) with the dynamics induced on generalized flag manifolds by the right invariant continuous-time flow generated by  $X$  (or the discrete-time flow generated by  $g$ ). We characterize the recurrent set and the finest Morse decomposition (including its stable sets) of these flows and show that its entropy always vanishes. We characterize the structurally stable ones and compute the Conley index of the attractor Morse component. When the nilpotent part of  $X$  is trivial, we compute the Conley indexes of all Morse components. Finally, we consider the dynamical aspects of linear differential equations with periodic coefficients in  $\mathfrak{g}$ , which can be regarded as an extension of the dynamics generated by an element  $X \in \mathfrak{g}$ . In this context, we generalize Floquet theory and extend the previous results to this case.

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# 1 Introduction

Let  $G$  be a linear group acting differentially on a manifold  $F$  and  $\mathfrak{g}$  its Lie algebra. We denote by  $g^t$ ,  $t \in \mathbb{T} = \mathbb{R}$  or  $\mathbb{Z}$ , the right invariant continuous-time flow generated by  $X \in \mathfrak{g}$  or the discrete-time flow generated by  $g \in G$ . More precisely, when  $\mathbb{T} = \mathbb{R}$ , we have that  $g^t = \exp(tX)$  and, when  $\mathbb{T} = \mathbb{Z}$ , we have that  $g^t$  is the  $t$ -iterate of  $g$ . When  $t = 1$  we just write  $g = g^1$ . Throughout this paper, we call  $g^t$  a linear flow. It induces a differentiable flow on  $F$  given by  $(t, x) \mapsto g^t x$ , where  $x \in F$  and  $t \in \mathbb{T} = \mathbb{Z}$  or  $\mathbb{R}$ . We call these flows linearly induced flows.

Take  $G = \text{Int}(\mathfrak{g})$ , where  $\mathfrak{g}$  is a semisimple Lie algebra. In [5] it is considered a continuous-time flow generated by real semisimple element  $H \in \mathfrak{g}$  acting on the flag manifolds of  $\mathfrak{g}$ : they show it is a Morse-Bott gradient flow, describe its fixed point set and their stable sets. In [7] it is analyzed a continuous-time flow generated by an element which is the sum of two commuting elements of  $\mathfrak{g}$ , one of which induces a gradient vector field and the other generates a one-parameter group of isometries. In the context of  $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$ , the articles [1, 2] study the discrete-time flow generated by an arbitrary element  $g \in \text{Sl}(n, \mathbb{R})$ : they characterize the structurally stable ones.

In this article, we study the dynamics of linearly induced flows  $g^t$ , for both continuous and discrete times, acting on a generalized flag manifold  $\mathbb{F}_\Theta$  of  $\mathfrak{g}$ . This context includes, for example, volume preserving and Hamiltonian linearly induced flows acting, respectively, on Grassmanian manifolds and on Grassmanian of isotropic subspaces, such as the Grassmanian of the Lagrangian subspaces. This dynamics is related with the Jordan decomposition of the flow  $g^t$ , which is defined in terms of the Jordan decomposition of  $X$  or  $g$  (see Section 3). We also consider the dynamical aspects of linear differential equations with periodic coefficients in  $\mathfrak{g}$ , which can be regarded as an extension of the dynamics generated by an element  $X \in \mathfrak{g}$ . In what follows we describe the structure of this article. We note that we recover, in this setting, the results of [14, 13] about flows in flag bundles with chain recurrent compact Hausdorff base.

In the preliminaries we recall some notions of Conley theory and of semisimple Lie theory, proving some useful results.

In Section 3, we recall the Jordan decomposition in  $G$  and  $\mathfrak{g}$  and show that the flow  $g^t$  can be written as a product of commuting flows  $g^t = e^t h^t u^t$ , where  $e^t$ ,  $h^t$ ,  $u^t$  are linear flows in  $G$  which are called, respectively, the elliptic, hyperbolic and unipotent components of  $g^t$ . We finish this section with a

result about the good behavior of the Jordan decomposition under a certain representation of  $G$  which is related to a natural immersion of a flag manifold into a projective space.

Section 4 is made up of various subsections. In the first one, we look at the linearly induced flow of  $g^t$  on a flag manifold as the restriction of a linearly induced flow on a projective space. Using the results of the appendix about dynamics on projective spaces and the results of [5] about the action of a real semisimple element  $H \in \mathfrak{g}$  on the flag manifolds, we generalize these results characterizing the recurrent and chain recurrent sets, the finest Morse decomposition, including its stable sets, in terms of the fixed points of the Jordan components. For example, we get the following result.

**Theorem 1.1** *The recurrent and chain recurrent sets of  $g^t$  in the flag manifold  $\mathbb{F}_\Theta$  are given, respectively, by*

$$\mathcal{R}(g^t) = \text{fix}_\Theta(h^t) \cap \text{fix}_\Theta(u^t) \quad \text{and} \quad \mathcal{R}_C(g^t) = \text{fix}_\Theta(h^t),$$

where  $\text{fix}_\Theta(h^t)$  and  $\text{fix}_\Theta(u^t)$  are the fixed points of these flows in  $\mathbb{F}_\Theta$ .

As a byproduct, we show that the entropy of these flows always vanishes. In Section 4.2 we define the conformal flows as the ones whose unipotent part in the Jordan decomposition is trivial, this is the kind of linear flow considered in [7]. For these flows, we compute the Conley indexes of all Morse components. We note that we can compute the Conley index of the attractor for every flow  $g^t$ , with no restrictions. We then introduce the regular flows, which are a particular case of the conformal flows. We show that they are dense in  $G$  or  $\mathfrak{g}$ , which implies Theorem 8.1 of [7] about the density of continuous-time conformal flows. Using this and the previous results we obtain the next result which generalizes results of [1, 2] obtained in the context of discrete-time flows generated by an arbitrary element  $g \in \text{Sl}(n, \mathbb{R})$ .

**Theorem 1.2** *The following conditions are equivalent:*

- (i)  $g^t$  is regular,
- (ii)  $g^t$  is Morse-Smale and
- (iii)  $g^t$  is structurally stable.

Finally, we consider the dynamical aspects of linear differential equations with periodic coefficients in  $\mathfrak{g}$ , which can be regarded as an extension of the dynamics generated by an element  $X \in \mathfrak{g}$ . In this context, we generalize Floquet theory and then extend the previous results to this case.

## 2 Preliminaries

### 2.1 Flows on topological spaces

Let  $\phi : \mathbb{T} \times X \rightarrow X$  be a continuous flow on a compact metric space  $(X, d)$ , with discrete  $\mathbb{T} = \mathbb{Z}$  or continuous  $\mathbb{T} = \mathbb{R}$  time. For a  $\phi^t$ -invariant set  $\mathcal{M} \subset X$ , we define its stable and unstable sets respectively as

$$\text{st}(\mathcal{M}) = \{x \in E : \omega(x) \subset \mathcal{M}\}, \quad \text{un}(\mathcal{M}) = \{x \in E : \omega^*(x) \subset \mathcal{M}\},$$

where  $\omega(x)$ ,  $\omega^*(x)$  are the limit sets of  $x$ . We denote by  $\mathcal{R}(\phi^t)$  the set of all recurrent points, that is

$$\mathcal{R}(\phi^t) = \{x \in X : x \in \omega(x)\},$$

and by  $\text{fix}(\phi^t)$  the set of all fixed points, that is

$$\text{fix}(\phi^t) = \{x \in X : \phi^t(x) = x, \text{ for all } t \in \mathbb{T}\}.$$

A linear flow  $\Phi^t$  on a vector bundle  $V$  is called normally hyperbolic if  $V$  can be written as a Whitney sum of their stable and unstable set and there exist a norm in  $V$  and constants  $\alpha, \beta > 0$  such that  $|\Phi^t(v)| < e^{-\alpha t}|v|$ , when  $v$  is in the stable set, and  $|\Phi^t(v)| < e^{\beta t}|v|$ , when  $v$  is in the unstable set. We say that a  $\phi^t$ -invariant set  $\mathcal{M} \subset X$  is normally hyperbolic if there exists a neighborhood of  $\mathcal{M}$  where the flow is conjugated to a normally hyperbolic linear flow restricted to some neighborhood of the null section.

We recall here the definitions and results related to the concept of chain recurrence and chain transitivity introduced in [4] (see also [12]). Take  $x, y \in X$ ,  $\varepsilon > 0$  and  $t \in \mathbb{T}$ . A  $(\varepsilon, t)$ -chain from  $x$  to  $y$  is a sequence of points  $\{x = x_1, \dots, x_{n+1} = y\} \subset X$  and a sequence of times  $\{t_1, \dots, t_n\} \subset \mathbb{T}$  such that  $t_i \geq t$  and  $d(\phi^{t_i}(x_i), x_{i+1}) < \varepsilon$ , for all  $i = 1, \dots, n$ .

Given a subset  $Y \subset X$  we write  $\Omega(Y, \varepsilon, t)$  for the set of all  $x$  such that there is a  $(\varepsilon, t)$ -chain from a point  $y \in Y$  to  $x$ . Also we put

$$\Omega^*(x, \varepsilon, t) = \{y \in X : x \in \Omega(y, \varepsilon, t)\}.$$

If  $Y \subset X$ , we write

$$\Omega(Y) = \bigcap \{\Omega(Y, \varepsilon, t) : \varepsilon > 0, t \in \mathbb{T}\}.$$

Also, for  $x \in X$  we write  $\Omega(x) = \Omega(\{x\})$  and define the relation  $x \preceq y$  if  $y \in \Omega(x)$ , which is transitive, closed and invariant by  $\phi^t$ , i.e., we have  $\phi^t(x) \preceq \phi^s(x)$  if  $x \preceq y$ , for all  $s, t \in \mathbb{T}$ . For every  $Y \subset X$  the set  $\Omega(Y)$  is invariant as well.

Define the relation  $x \sim y$  if  $x \preceq y$  and  $y \preceq x$ . Then  $x \in X$  is said to be chain recurrent if  $x \sim x$ . We denote by  $\mathcal{R}_C(\phi^t)$  the set of all chain recurrent points. It is easy to see that the restriction of  $\sim$  to  $\mathcal{R}_C(\phi^t)$  is an equivalence relation. An equivalence class of  $\sim$  is called a *chain transitive component* or a *chain component*, for short.

Now we prove two results which will be used further on.

**Lemma 2.1** *Let  $e^t$  be a flow of  $X$  such that  $e^t$  is an isometry for all  $t \in \mathbb{T}$ . Then, for each  $T \in \mathbb{T}$  and each  $x \in X$ , there exists a sequence  $n_k \rightarrow \infty$  such that  $g^{n_k T} x \rightarrow x$ .*

**Proof:** By the compactness of  $X$ , we have that the sequence  $e^{nT}x$  has a convergent subsequence. Thus, given  $\varepsilon > 0$  and  $L > 0$ , there exist  $m, k \in \mathbb{N}$  such that  $m - k > L$  and

$$d(e^{(m-k)T}x, x) = d(e^{mT}x, e^{kT}x) < \varepsilon.$$

Hence there exists a sequence  $n_k \rightarrow \infty$  such that  $g^{n_k T}x \rightarrow x$ . □

**Lemma 2.2** *Let  $e^t, u^t$  be commuting flows of  $X$ ,  $t \in \mathbb{T}$ . Assume that  $e^t$  is an isometry for all  $t \in \mathbb{T}$  and that for each  $x \in X$  there exists  $y \in X$  such that the omega and alpha limits of  $x$  by  $u^t$  are precisely  $y$ . Then the composition  $e^t u^t$  is a chain recurrent flow.*

**Proof:** Fix  $x \in X$ . Given  $\varepsilon > 0$  and  $t_0 > 0$  we will construct an  $(\varepsilon, t_0)$ -chain from  $x$  to  $x$ . By the assumption on  $u$ , there exists  $y \in X$  and  $t_1 > t_0$  such that

$$u^t(x), u^{-t}(x) \in B(y, \varepsilon/2),$$

for all  $t > t_1$ . Taking  $t > t_1$ , it follows that the points  $\{x, u^{-t}(x), x\}$  and times  $\{t, t\}$  define an  $(\varepsilon, t_0)$ -chain of  $u$ , since

$$d(u^t(x), u^{-t}(x)) < \varepsilon \quad \text{and} \quad d(u^t u^{-t}(x), x) = 0 < \varepsilon.$$

Now, since the isometry  $e$  is recurrent (see Lemma 2.1), there exists  $t > t_1$  such that  $d(e^{2t}(x), x) < \varepsilon$ . Thus the points  $\{x, e^t u^{-t}(x), x\}$  and times  $\{t, t\}$

define an  $(\varepsilon, t_0)$ -chain of  $eu$ . In fact, using the commutativity of  $e$  and  $u$  and using that  $e$  is an isometry, we have

$$d((eu)^t(x), e^t u^{-t}(x)) = d(u^t(x), u^{-t}(x)) < \varepsilon,$$

by the above construction. Finally, using again the commutativity of  $e$  and  $u$  we have that

$$d((eu)^t e^t u^{-t}(x), x) = d(e^{2t}(x), x) < \varepsilon,$$

by the choice of  $t$ . □

Now we relate Morse decompositions to chain transitivity. First let us recall that a finite collection of disjoint subsets  $\{\mathcal{M}_1, \dots, \mathcal{M}_n\}$  defines a Morse decomposition when

- (i) each  $\mathcal{M}_i$  is compact and  $\phi^t$ -invariant,
- (ii) for all  $x \in X$  we have  $\omega(x), \omega^*(x) \subset \bigcup_i \mathcal{M}_i$ ,
- (iii) if  $\omega(x), \omega^*(x) \subset \mathcal{M}_j$  then  $x \in \mathcal{M}_j$ .

Each set  $\mathcal{M}_i$  of a Morse decomposition is called a Morse component. If  $\{\mathcal{M}_1, \dots, \mathcal{M}_n\}$  is a Morse decomposition of  $X$ , then it is immediate that  $X$  decomposes as the disjoint union of stable sets  $\text{st}(\mathcal{M}_i)$ .

The finest Morse decomposition is a Morse decomposition which is contained in every other Morse decomposition. The existence of a finest Morse decomposition of a flow is equivalent to the finiteness of the number of chain components (see [12], Theorem 3.15). In this case, each Morse component is a chain transitive component and vice-versa. We say that the flow  $\phi^t$  is normally hyperbolic if there exists the finest Morse decomposition and their Morse components are normally hyperbolic.

## 2.2 Semi-simple Lie theory

For the theory of semisimple Lie groups and their flag manifolds we refer to Duistermaat-Kolk-Varadarajan [5], Helgason [6] and Warner [17]. To set notation let  $\mathfrak{g}$  be a semisimple Lie algebra and  $G = \text{Int}(\mathfrak{g}) \subset \text{Gl}(\mathfrak{g})$  acting in  $\mathfrak{g}$  canonically. We identify throughout the Lie algebra of  $G$  with  $\mathfrak{g}$ , that is, we write  $g = \exp(X)$  to mean  $g = e^{\text{ad}(X)}$ , where  $X \in \mathfrak{g}$ . Thus, for  $g \in G$  and

$X \in X$ , it follows that  $g \exp(X) g^{-1} = \exp(gX)$ . Note that if  $\tilde{G}$  is a connected Lie group with Lie algebra  $\mathfrak{g}$ , then  $\text{Ad}(\tilde{G}) = \text{Int}(\mathfrak{g}) = G$ . It follows that the adjoint action of  $\tilde{G}$  in  $\mathfrak{g}$  is the canonical action of  $G$  in  $\mathfrak{g}$ .

Fix a Cartan involution  $\theta$  of  $\mathfrak{g}$  with Cartan decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ . The form  $B_\theta(X, Y) = -\langle X, \theta Y \rangle$ , where  $\langle \cdot, \cdot \rangle$  is the Cartan-Killing form of  $\mathfrak{g}$ , is an inner product.

Fix a maximal abelian subspace  $\mathfrak{a} \subset \mathfrak{s}$  and a Weyl chamber  $\mathfrak{a}^+ \subset \mathfrak{a}$ . We let  $\Pi$  be the set of roots of  $\mathfrak{a}$ ,  $\Pi^+$  the positive roots corresponding to  $\mathfrak{a}^+$ ,  $\Sigma$  the set of simple roots in  $\Pi^+$  and  $\Pi^- = -\Pi^+$  the negative roots. The Iwasawa decomposition of the Lie algebra  $\mathfrak{g}$  reads  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}^\pm$  with  $\mathfrak{n}^\pm = \sum_{\alpha \in \Pi^\pm} \mathfrak{g}_\alpha$  where  $\mathfrak{g}_\alpha$  is the root space associated to  $\alpha$ . As to the global decompositions of the group we write  $G = KS$  and  $G = KAN^\pm$  with  $K = \exp \mathfrak{k}$ ,  $S = \exp \mathfrak{s}$ ,  $A = \exp \mathfrak{a}$  and  $N^\pm = \exp \mathfrak{n}^\pm$ .

The Weyl group  $W$  associated to  $\mathfrak{a}$  is the finite group generated by the reflections over the root hyperplanes  $\alpha = 0$  in  $\mathfrak{a}$ ,  $\alpha \in \Pi$ . For each  $w \in W$  and  $\alpha \in \Pi$  we define  $w^*\alpha(H) = \alpha(w^{-1}H)$ , for all  $H \in \mathfrak{a}$ . We have that  $w^*\alpha \in \Pi$  and that this is a transitive action of  $W$  on  $\Pi$ . The maximal involution  $w^-$  of  $W$  is the (only) element of  $W$  which is such that  $(w^-)^*\Sigma = -\Sigma$ .

Given a subset of simple roots  $\Theta \subset \Sigma$ , let

$$\mathfrak{a}_\Theta = \{H \in \mathfrak{a} : \alpha(H) = 0, \alpha \in \Theta\}$$

and put  $A_\Theta = \exp(\mathfrak{a}_\Theta)$ . Let also

$$\mathfrak{n}(\Theta)^\pm = \sum_{\alpha \in \langle \Theta \rangle \cap \Pi^\pm} \mathfrak{g}_\alpha \quad \text{and} \quad \mathfrak{n}_\Theta^\pm = \sum_{\alpha \in \Pi^\pm - \langle \Theta \rangle} \mathfrak{g}_\alpha$$

and put  $N_\Theta^\pm = \exp(\mathfrak{n}_\Theta^\pm)$ . The subset  $\Theta$  singles out the subgroup  $W_\Theta$  of the Weyl group which acts trivially on  $\mathfrak{a}_\Theta$ .

The standard parabolic subalgebra of type  $\Theta \subset \Sigma$  with respect to chamber  $\mathfrak{a}^+$  is defined by

$$\mathfrak{p}_\Theta = \mathfrak{n}^-(\Theta) \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}^+.$$

Let  $p$  the dimension of  $\mathfrak{p}_\Theta$  and denote the grassmanian of  $p$ -dimensional subspaces of  $\mathfrak{g}$  by  $\text{Gr}_p(\mathfrak{g})$ . The flag manifold of type  $\Theta$  is the orbit  $\mathbb{F}_\Theta = G\mathfrak{p}_\Theta \subset \text{Gr}_p(\mathfrak{g})$ , with base point  $b_\Theta = \mathfrak{p}_\Theta$ , which identifies with the homogeneous space  $G/P_\Theta$ . Since the center of  $G$  normalizes  $\mathfrak{p}_\Theta$ , the flag manifold depends only on the Lie algebra  $\mathfrak{g}$  of  $G$ . The empty set  $\Theta = \emptyset$  gives the maximal flag manifold  $\mathbb{F} = \mathbb{F}_\emptyset$  with basepoint  $b = b_\emptyset$ .

For  $H \in \mathfrak{a}$  we denote by  $Z_H, K_H, W_H$ , the centralizer of  $H$  in  $G, K, W$ , respectively, i.e, the elements in those groups which fix  $H$ . Note that  $g \in G$  centralizes  $H$  if and only if it commutes with  $\text{ad}(H)$ . In fact, this follow from  $\text{ad}(gH) = g\text{ad}(H)g^{-1}$  and the injectivity of  $\text{ad}$ . When  $H \in \text{cl}\mathfrak{a}^+$  we put

$$\Theta(H) = \{\alpha \in \Sigma : \alpha(H) = 0\}.$$

An element  $H \in \text{cl}\mathfrak{a}^+$  induces a vector field  $\tilde{H}$  on a flag manifold  $\mathbb{F}_\Theta$  with flow  $\exp(tH)$ . This is a gradient vector field with respect to a given Riemannian metric on  $\mathbb{F}_\Theta$  (see [5], Section 3). The connected sets of fixed point of this flow are given by

$$\text{fix}_\Theta(H, w) = Z_H w b_\Theta = K_H w b_\Theta,$$

so that they are in bijection with the cosets in  $W_H \backslash W / W_\Theta$ . Each  $w$ -fixed point connected set has stable manifold given by

$$\text{st}_\Theta(H, w) = N_{\Theta(H)}^- \text{fix}_\Theta(H, w),$$

whose union gives the Bruhat decomposition of  $\mathbb{F}_\Theta$ :

$$\mathbb{F}_\Theta = \coprod_{W_H \backslash W / W_\Theta} \text{st}_\Theta(H, w).$$

The unstable manifold is

$$\text{un}_\Theta(H, w) = N_{\Theta(H)}^+ \text{fix}_\Theta(H, w).$$

We note that both  $\text{st}_\Theta(H, 1)$  and  $\text{un}_\Theta(H, w^-)$  are open and dense in  $\mathbb{F}_\Theta$ . Since the centralizer  $Z_H$  of  $H$  leaves  $\text{fix}_\Theta(H, w)$  invariant and normalizes both  $N_{\Theta(H)}^-$  and  $N_{\Theta(H)}^+$ , it follows  $\text{st}_\Theta(H, w)$  and  $\text{un}_\Theta(H, w)$  are  $Z_H$ -invariant. We note that these fixed points and (un)stable sets remain the same if  $H$  is replaced by some  $H' \in \text{cl}\mathfrak{a}^+$  such that  $\Theta(H') = \Theta(H)$ .

We note that, since the spectrum of  $\text{ad}(H)$  in  $\mathfrak{g}_\alpha$  is  $\alpha(H)$ , it follows that the spectrum of  $h = \exp(H)$  in  $\mathfrak{g}_\alpha$  is  $e^{\alpha(H)}$ .

We conclude with a useful lemma about the decomposition semisimple elements. We say that  $X \in \mathfrak{g}$  is semisimple if  $\text{ad}(X)$  is diagonalizable over  $\mathbb{C}$  and that  $g \in G$  is semisimple if  $g$  is diagonalizable over  $\mathbb{C}$ .

**Lemma 2.3** *We have that*



- (i) If  $X \in \mathfrak{g}$  is semisimple, then there exists a Cartan decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$  such that  $X = E + H$  where  $H \in \mathfrak{s}$  and  $E \in \mathfrak{k}_H$ .
- (ii) If  $g \in G$  is semisimple, then there exists a Cartan decomposition  $G = KS$  such that  $g = eh$ , where  $h = \exp(H)$ ,  $H \in \mathfrak{s}$  and  $e \in K_H$ .

**Proof:** For item (i), since  $X$  is semisimple, there exists a Cartan subalgebra  $\mathfrak{j}$  such that  $X \in \mathfrak{j}$  (see the proof of Proposition 1.3.5.4, p.105 of [17]). By Proposition 1.3.1.1, p.89 of [17], there exists a Cartan involution  $\theta$  such that  $\mathfrak{j}$  is  $\theta$ -invariant. Thus we have that

$$\mathfrak{j} = (\mathfrak{j} \cap \mathfrak{k}) \oplus (\mathfrak{j} \cap \mathfrak{s}).$$

Writing  $X = E + H$ , with  $E \in \mathfrak{j} \cap \mathfrak{k}$  and  $H \in \mathfrak{j} \cap \mathfrak{s}$ , we have that  $E$  and  $H$  commute, since  $\mathfrak{j}$  is abelian.

For item (ii), since  $g$  is semisimple, there exists a Cartan subgroup  $J$  such that  $g \in J$  (since the centralizer of  $g$  in  $\mathfrak{g}$  contains a Cartan subalgebra, see the proof of Proposition 1.4.3.2, p.120 of [17]). Denote by  $\mathfrak{j}$  the associated Cartan subalgebra. By Proposition 1.3.1.1, p.89 of [17], there exists a Cartan involution  $\theta$  such that  $\mathfrak{j}$  is  $\theta$ -invariant. Thus, by Proposition 1.4.1.2, p.109 of [17], we have that

$$J = (J \cap K)(\exp(\mathfrak{j} \cap \mathfrak{s})).$$

Writing  $g = eh$ , with  $e \in J \cap K$  and  $h = \exp(H)$ , where  $H \in \mathfrak{j} \cap \mathfrak{s}$ . Since  $J$  centralizes  $\mathfrak{j}$ , it follows that  $e$  and  $\text{ad}(H)$  commute, showing that  $e \in K_H$ .  $\square$

### 3 Jordan decomposition

In this section we recall the additive and the multiplicative Jordan decompositions. Let  $V$  be a finite dimensional vector space.

If  $X \in \text{gl}(V)$ , then we can write  $X = E + H + N$ , where  $E \in \text{gl}(V)$  is semisimple with imaginary eigenvalues,  $H \in \text{gl}(V)$  is diagonalizable in  $V$  with real eigenvalues and  $N \in \text{gl}(V)$  is nilpotent. The linear maps  $E$ ,  $H$  and  $N$  commute, are unique and called, respectively, the *elliptic*, the *hyperbolic*, and the *nilpotent* components of the additive Jordan decomposition of  $X$  (see Section ? of [?]).

If  $g \in \text{Gl}(V)$ , then we can write  $g = eh$ , where  $e \in \text{Gl}(V)$  is an isometry relative to some appropriate inner product,  $h \in \text{Gl}(V)$  is diagonalizable in

$V$  with positive eigenvalues and  $u \in \text{Gl}(V)$  is the exponential of a nilpotent linear map. The linear maps  $e$ ,  $h$  and  $u$  commute, are unique and called, respectively, the *elliptic*, the *hyperbolic* and the *unipotent* components of the multiplicative Jordan decomposition of  $g$  (see Lemma IX.7.1 p.430 of [6]). We denote by  $\log h$  the matrix given by the logarithm of the diagonal elements of  $h$  in the Jordan basis. Writing  $g$ ,  $e$ ,  $h$ ,  $u$  in the Jordan basis, we see that they commute with  $\log h$ .

Take  $\mathfrak{g}$  a semisimple Lie algebra. We say that  $X = E + H + N$ , where  $E, H, N \in \mathfrak{g}$ , is the Jordan decomposition of  $X$  in  $\mathfrak{g}$  if  $\text{ad}(X) = \text{ad}(E) + \text{ad}(H) + \text{ad}(N)$  is the additive Jordan decomposition of  $\text{ad}(X)$  in  $\text{gl}(\mathfrak{g})$ . In this case,  $E$ ,  $H$  and  $N$  commute, are unique and called, respectively, the *elliptic*, the *hyperbolic*, and the *nilpotent* components of  $X$ .

We note that the conjugate of a Jordan decomposition is the Jordan decomposition of the conjugate. Now we prove the following useful result.

**Lemma 3.1** *Let  $G = \text{int}(\mathfrak{g})$ , where  $\mathfrak{g}$  is a semisimple Lie algebra. Then we have that*

- (i) *For each  $X \in \mathfrak{g}$ , there exists the Jordan decomposition  $X = E + H + N$ . Furthermore, there exists an Iwasawa decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}^+$  such that  $E \in \mathfrak{k}_H$  and  $H \in \text{cl } \mathfrak{a}^+$ .*
- (ii) *For each  $g \in G$ , its multiplicative Jordan components  $e, h, u$  lie in  $G$ . Moreover, there exist a unique  $H \in \mathfrak{g}$  such that  $\log h = \text{ad}(H)$  and an Iwasawa decomposition  $G = KAN$  such that  $e \in K_H$  and  $H \in \text{cl } \mathfrak{a}^+$ .*

**Proof:** For item (i), by Proposition 1.3.5.1, p.104 of [17], there exists a unique decomposition  $X = S + N$ , where  $S, N \in \mathfrak{g}$  commute,  $S$  is semisimple and  $\text{ad}(N)$  is nilpotent. By Lemma 2.3, there exists an Cartan decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$  such that  $S = E + H$ , where  $H \in \mathfrak{s}$  and  $E \in \mathfrak{k}_H$ . This is the additive Jordan decomposition of  $S$  in  $\mathfrak{g}$ , since  $\text{ad}(E)$  is  $B_\theta$ -anti-symmetric and  $\text{ad}(H)$  is  $B_\theta$ -symmetric. It remains to show that  $\text{ad}(E)$  and  $\text{ad}(H)$  commute with  $\text{ad}(N)$ . We first note that  $u = I + \text{ad}(N)$  is invertible and that  $Y \in \text{gl}(\mathfrak{g})$  commutes with  $\text{ad}(N)$  if and only if  $Y$  commutes with  $u$ . In fact, we have that

$$Y + Y\text{ad}(N) = Yu = uY = Y + \text{ad}(N)Y.$$

It follows that  $u$  commutes with  $\text{ad}(X)$ . In order to show that  $\text{ad}(E)$  and  $\text{ad}(H)$  commute with  $u$ , we write

$$\text{ad}(E) + \text{ad}(H) + \text{ad}(N) = u\text{ad}(X)u^{-1} = u\text{ad}(E)u^{-1} + u\text{ad}(H)u^{-1} + \text{ad}(N).$$

By the uniqueness of the additive Jordan decomposition in  $\mathfrak{gl}(\mathfrak{g})$ , we have that  $\text{ad}(E) = u\text{ad}(E)u^{-1}$  and  $\text{ad}(H) = u\text{ad}(H)u^{-1}$ . Since  $H \in \mathfrak{s}$ , we can choose an Iwasawa decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}^+$  such that  $E \in \mathfrak{k}_H$  and  $H \in \text{cl } \mathfrak{a}^+$ .

For item (ii), by Proposition 1.4.3.3, p.120 of [17], there exists a unique decomposition  $g = \widehat{s}\widehat{u}$ , where  $\widehat{s}, \widehat{u} \in G$  commute,  $\widehat{s}$  is semisimple and  $\widehat{u}$  is the exponential of a nilpotent linear map. By Lemma 2.3, there exists an Cartan decomposition  $G = KS$  such that  $\widehat{s} = \widehat{e}\widehat{h}$ , where  $\widehat{h} = \exp(\widehat{H})$ ,  $\widehat{H} \in \mathfrak{s}$  and  $\widehat{e} \in K_{\widehat{H}}$ . This is the multiplicative Jordan decomposition of  $g$ , since  $\widehat{e}$  is a  $B_\theta$ -isometry and  $\widehat{h}$  is  $B_\theta$ -positive. In order to show that  $\widehat{e}$  and  $\widehat{h}$  commute with  $\widehat{u}$ , one can proceed as in Lemma IX.7.1 p.431 of [6]. By the uniqueness of the multiplicative Jordan decomposition in  $\text{Gl}(\mathfrak{g})$ , it follows that  $e = \widehat{e}$ ,  $h = \widehat{h}$  and  $u = \widehat{u}$ , showing that the multiplicative Jordan components of  $g$  lie in  $G$ . By the proof of Lemma IX.7.3 item (i) p.431 of [6], we have that  $\log h$  lies in the Lie algebra of  $G$  and thus there exists a unique  $H \in \mathfrak{g}$  such that  $\log h = \text{ad}(H)$ , since  $\text{ad}$  is injective. Since both  $\text{ad}(H)$  and  $\text{ad}(\widehat{H})$  commute with  $e = \widehat{e}$ , it follows that  $\text{ad}(H)$  and  $\text{ad}(\widehat{H})$  can be diagonalized in the same basis. Since  $e^{\text{ad}(H)} = h = e^{\text{ad}(\widehat{H})}$  and using the injectivity of  $\text{ad}$ , it follows that  $H = \widehat{H} \in \mathfrak{s}$ . Thus we can choose an Iwasawa decomposition  $G = KAN$  such that  $e \in K_H$  and  $H \in \text{cl } \mathfrak{a}^+$ .  $\square$

Let  $G$  be a linear group. Now we define the Jordan decomposition of a linear flow  $g^t$  in  $G$ ,  $t \in \mathbb{T}$ . If  $\mathbb{T} = \mathbb{R}$  then  $g^t = e^{tX}$ ,  $X \in \mathfrak{gl}(V)$  and we can use the additive Jordan decomposition  $X = E + H + N$  to write  $g^t = e^t h^t u^t$ , where  $e^t = e^{tE}$ ,  $h^t = e^{tH}$  and  $u^t = e^{tN}$ . If  $\mathbb{T} = \mathbb{Z}$  we can use the multiplicative Jordan decomposition to write  $g^t = e^t h^t u^t$  for each  $t \in \mathbb{T}$ . It follows that in both cases the linear flows  $g^t$ ,  $e^t$ ,  $h^t$ ,  $u^t$  commute.

Now take  $G = \text{Int}(\mathfrak{g})$ , where we identify the Lie algebra of  $G$  with  $\mathfrak{g}$  (see Section 2.2). Let  $g^t \in G$ , for all  $t \in \mathbb{T}$ . By Lemma 3.1, each Jordan component  $e^t, h^t, u^t$  of  $g^t$  also lies in  $G$ , for all  $t \in \mathbb{T}$ . If  $\mathbb{T} = \mathbb{Z}$  then this is immediate. When  $\mathbb{T} = \mathbb{R}$ , then  $g^t = \exp(tX)$ , where  $X \in \mathfrak{g}$ . Thus we can use the Jordan decomposition  $X = E + H + N$  to write  $g^t = e^t h^t u^t$ , where  $e^t = \exp(tE)$ ,  $h^t = \exp(tH)$  and  $u^t = \exp(tN)$ . In both continuous and discrete time cases, we also have that each Jordan components of the flow  $g^t$  lie in  $Z_H$ , where  $H$  is given by Lemma 3.1, when  $\mathbb{T} = \mathbb{Z}$ .

Let  $\rho : G \rightarrow \text{Gl}(V)$  be a finite dimensional representation, where  $d_1\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$  is its infinitesimal representation. When  $t \in \mathbb{Z}$ , it is immediate

that  $\rho(g^t) = \rho(g)^t$ . When  $t \in \mathbb{R}$ , we have that  $g^t = \exp(tX)$ , for  $X \in \mathfrak{g}$ . Denoting  $\rho(g)^t = \exp(td_1\rho X)$ , it follows also that  $\rho(g^t) = \rho(g)^t$ .

Now we consider the behavior of the Jordan decomposition with respect to the canonical representation of the general linear group  $\mathrm{Gl}(L)$  in  $\mathrm{Gl}(\bigwedge^p L)$ , where  $L$  be a finite dimensional vector space, given by

$$\rho(g)v_1 \wedge \cdots \wedge v_p = gv_1 \wedge \cdots \wedge gv_p.$$

**Lemma 3.2** *For the the canonical representation of the general linear group  $\mathrm{Gl}(L)$  in  $\mathrm{Gl}(\bigwedge^p L)$ , we have that*

- i) *Take  $X \in \mathfrak{gl}(L)$ . If  $X$  is elliptic (resp. hyperbolic, nilpotent), then  $d_1\rho X$  is elliptic (resp. hyperbolic, nilpotent). In particular, if  $X = E + H + N$  is the additive Jordan decomposition of  $X$ , then  $d_1\rho X = d_1\rho E + d_1\rho H + d_1\rho N$  is the additive Jordan decomposition of  $d_1\rho X$ .*
- ii) *Take  $g \in \mathrm{Gl}(L)$ . If  $g$  is elliptic (resp. hyperbolic, unipotent), then  $\rho(g)$  is elliptic (resp. hyperbolic, unipotent). In particular, if  $g = ehu$  is the Jordan decomposition of  $g$ , then  $\rho(g) = \rho(e)\rho(h)\rho(u)$  is the Jordan decomposition of  $\rho(g)$ .*
- iii) *If  $e^th^tu^t$  is the Jordan decomposition of  $g^t$ , then  $\rho(e)^t\rho(h)^t\rho(u)^t$  is the Jordan decomposition of  $\rho(g)^t$ .*

**Proof:** First observe that the complexification of wedge product of  $L$  is equal to the wedge product of the complexification of  $L$ , that is  $(\bigwedge^p L)_{\mathbb{C}} = \bigwedge^p L_{\mathbb{C}}$ . In fact, it is immediate that  $\bigwedge^p L_{\mathbb{C}} \subset (\bigwedge^p L)_{\mathbb{C}}$  and that both have the same dimension. Note also that

$$(d_1\rho X)v_1 \wedge \cdots \wedge v_p = \sum_{i=1}^p v_1 \wedge \cdots \wedge Xv_i \wedge \cdots \wedge v_p.$$

We claim that

$$(d_1\rho X)^m v_1 \wedge \cdots \wedge v_p = \sum_i w_{i1} \wedge \cdots \wedge w_{ip},$$

where  $w_{ij} \in N^{m_{ij}}(L)$  such that  $\sum_{j=1}^p m_{ij} = m$ . For  $m = 0$  this is immediate. By induction on  $m$

$$(d_1\rho X)^{m+1} v_1 \wedge \cdots \wedge v_p = (d_1\rho X) \sum_i w_{i1} \wedge \cdots \wedge w_{ip} =$$

$$= \sum_i \sum_{j=1}^p w_{i1} \wedge \cdots \wedge Xw_{ij} \wedge \cdots \wedge w_{ip}$$

Since  $w_{ij} \in N^{m_{ij}}(L)$  it follows that  $Xw_{ij} \in N^{m_{ij}+1}(L)$ .

For item (i), taking  $X$  nilpotent then there exists  $l$  such that  $X^l = 0$ . From the above claim, it follows that  $(d_1\rho X)^{pl} = 0$ . In fact,

$$(d_1\rho X)^{pl} v_1 \wedge \cdots \wedge v_p = \sum_i w_{i1} \wedge \cdots \wedge w_{ip},$$

where  $w_{ij} \in N^{m_{ij}}(L)$  such that  $\sum_{j=1}^p m_{ij} = pl$ . Thus, for each  $i$  there exists  $j$  such that  $m_{ij} \geq l$ . Therefore  $w_{ij} = 0$ , which implies that  $w_{i1} \wedge \cdots \wedge w_{ip} = 0$ , for all  $i$ . Now taking  $X$  elliptic, there exists a  $\mathbb{C}$ -basis  $v_1, \dots, v_n$  of  $L_{\mathbb{C}}$  such that  $Xv_k = z_k v_k$ , where  $z_k$  is purely imaginary. Then

$$\{v_{i_1} \wedge \cdots \wedge v_{i_p} : \text{where } 1 \leq i_1 < \cdots < i_p \leq n\}$$

is a  $\mathbb{C}$ -basis of  $\bigwedge^p L_{\mathbb{C}}$  such that

$$(d_1\rho X)v_{i_1} \wedge \cdots \wedge v_{i_p} = (z_{i_1} + \cdots + z_{i_p})v_{i_1} \wedge \cdots \wedge v_{i_p}.$$

This implies that  $d_1\rho X$  is elliptic, since  $z_{i_1} + \cdots + z_{i_p}$  is purely imaginary. The hyperbolic case is analogous.

For item (ii), taking  $g$  unipotent, then  $g = e^N$  with  $N$  nilpotent so that  $\rho(g) = e^{d_1\rho(N)}$  is unipotent, by using item (i). Now taking  $g$  is elliptic, there exists a  $\mathbb{C}$ -basis  $v_1, \dots, v_n$  of  $L_{\mathbb{C}}$  such that  $gv_k = z_k v_k$ , where  $z_k \in \mathbb{C}$  with  $|z_k| = 1$ . Then

$$\{v_{i_1} \wedge \cdots \wedge v_{i_p} : \text{where } 1 \leq i_1 < \cdots < i_p \leq n\}$$

is a  $\mathbb{C}$ -basis of  $\bigwedge^p L_{\mathbb{C}}$  such that

$$\rho(g)v_{i_1} \wedge \cdots \wedge v_{i_p} = (z_{i_1} \cdots z_{i_p})v_{i_1} \wedge \cdots \wedge v_{i_p}.$$

This implies that  $\rho(g)$  is elliptic, since  $|z_{i_1} \cdots z_{i_p}| = 1$ . The hyperbolic case is analogous.

Item (iii) follows immediately from the previous items.  $\square$

We recall the well known Plücker embedding, which is given by

$$i : \text{Gr}_p(L) \rightarrow \mathbb{P}(\bigwedge^p L), \quad P \mapsto [v_1 \wedge \cdots \wedge v_p],$$

where  $\{v_1, \dots, v_p\}$  is a basis of  $P$ . This embedding has the following equivariance property

$$i(gP) = \rho(g)i(P)$$

where  $\rho$  is the canonical representation presented in Lemma 3.2. If  $g^t$  is a linearly induced flow it follows that

$$i(g^t P) = \rho(g)^t i(P).$$

## 4 Dynamics in flag manifolds

In this section, we relate the Jordan decomposition of  $g^t$  in  $G = \text{Int}(\mathfrak{g})$  to the dynamics of the induced linear flow  $g^t$  on the flag manifolds of  $\mathfrak{g}$ , where  $\mathfrak{g}$  is a semisimple Lie algebra. The main results of the section deals with the characterization of the recurrent set and the finest Morse decomposition in terms of the fixed points of the Jordan components.

Recall that, as seen in Section 3, when  $g^t \in G$  then each multiplicative Jordan component  $e^t, h^t, u^t$  of  $g^t$  lies in  $Z_H$ , where  $H \in \mathfrak{g}$  is such that  $\log h = \text{ad}(H)$ . Furthermore, there exists a Weyl chamber  $\mathfrak{a}^+$  such that  $H \in \text{cl } \mathfrak{a}^+$ .

Let  $\Theta \subset \Sigma$ . It follows that  $e^t, h^t$  and  $u^t$  induce flows in the flag manifold  $\mathbb{F}_\Theta$ . If  $p = \dim(\mathfrak{p}_\Theta)$ , we know that  $\mathbb{F}_\Theta \subset \text{Gr}_p(\mathfrak{g})$ , so we can restrict the Plücker embedding (see Section 3) to  $\mathbb{F}_\Theta$  and get an embedding  $i : \mathbb{F}_\Theta \rightarrow \mathbb{P}V$ , where  $V = \bigwedge^p \mathfrak{g}$ . Since  $\mathbb{F}_\Theta$  is  $G$ -invariant, we have the following equivariance property

$$i(gx) = \rho(g)i(x), \quad x \in \mathbb{F}_\Theta,$$

where  $\rho : G \rightarrow \text{Gl}(V)$  is the restriction to  $G$  of the canonical representation presented in Lemma 3.2.

### 4.1 Recurrence, chain recurrence and entropy

The next proposition shows that the fixed points of the hyperbolic part of  $g^t$  are Morse components for the flow  $g^t$  in the flag manifold. This result is proved by using Proposition A.5, which is in fact a particular case when the flag manifold is the projective space.

**Proposition 4.1** *Let  $g^t$  be a flow on  $\mathbb{F}_\Theta$ . The set*

$$\{\text{fix}_\Theta(H, w) : w \in W_H \backslash W/W_\Theta\}$$

is a Morse decomposition for  $g^t$ . Furthermore, the stable and unstable sets of  $\text{fix}_\Theta(H, w)$  are given by

$$\text{st}(\text{fix}_\Theta(H, w)) = \text{st}_\Theta(H, w) \quad \text{and} \quad \text{un}(\text{fix}_\Theta(H, w)) = \text{un}_\Theta(H, w),$$

**Proof:** Since  $\text{fix}_\Theta(H, w) = Z_H w \mathfrak{p}_\Theta$  and since  $g^t \in Z_H$ , it follows that  $\text{fix}_\Theta(H, w)$  is  $g^t$ -invariant.

Now we show that  $\text{st}_\Theta(H, w)$  is the stable set of  $\text{fix}_\Theta(H, w)$ . By the Bruhat decomposition of  $\mathbb{F}_\Theta$ , it is enough to show that  $\text{st}_\Theta(H, w)$  is contained in the stable set of  $\text{fix}_\Theta(H, w)$ . Let  $x \in \text{st}_\Theta(H, w)$ , then  $x = \exp(Y) l w b_\Theta$ , where  $Y \in \mathfrak{n}_{\Theta(H)}^-$  and  $l \in Z_H$ . Then

$$g^t x = g^t \exp(Y) g^{-t} g^t l w b_\Theta = \exp(g^t Y) g^t l w b_\Theta,$$

where  $g^t l w b_\Theta \in \text{fix}_\Theta(H, w)$ , since  $g^t l \in Z_H$ . Now we show that  $g^t Y \rightarrow 0$ . This follows by Lemma A.3, since the spectral radius of the restriction of  $g$  to  $\mathfrak{n}_{\Theta(H)}^-$  is smaller than 1. In fact, by the Jordan decomposition,  $r(g)$  is given by the greatest eigenvalue of its hyperbolic component, which is given by the restriction of  $h$  to  $\mathfrak{n}_{\Theta(H)}^-$ . These eigenvalues are given by  $e^{-\alpha(H)}$ , where  $\alpha \in \Pi^+$  with  $\alpha(H) > 0$ , so that  $r(g) < 1$ . Now if  $g^{t_j} x \rightarrow y$  then  $g^{t_j} l w b_\Theta \rightarrow y$ , so that  $y$  lies in the closed subset  $\text{fix}_\Theta(H, w)$ .

For the unstable set we proceed analogously. It follows that

$$\text{fix}_\Theta(h^t) = \bigcup_{w \in W} \text{fix}_\Theta(H, w)$$

contains all the alpha and omega limit sets. In order to show that the set  $\{\text{fix}_\Theta(H, w) : w \in W_H \setminus W/W_\Theta\}$  is a Morse decomposition for  $g^t$  it is enough to prove that if  $\omega(x), \omega^*(x) \subset \text{fix}_\Theta(H, w)$ , then  $x \in \text{fix}_\Theta(H, w)$ . First recall that

$$\begin{aligned} i(\text{fix}_\Theta(h^t)) &= \text{fix}(\rho(h)^t) \cap i(\mathbb{F}_\Theta), \\ i(\omega(x)) &= \omega(i(x)) \quad \text{and} \quad i(\omega^*(x)) = \omega^*(i(x)), \end{aligned}$$

where  $i$  is the Plücker embedding. By hypothesis  $\omega(i(x)), \omega^*(i(x))$  are contained in the connected set  $i(\text{fix}_\Theta(H, w))$  of  $\text{fix}(\rho(h)^t)$ , so they lie in the same connected component of  $\text{fix}(\rho(h)^t)$  which is given by an eigenspace of  $\rho(h)$ . Using Lemmas 3.2 and A.4, it follows that  $i(x) \subset \text{fix}(\rho(h)^t)$ , which shows that  $x \in \text{fix}_\Theta(h^t)$ . Then there exists  $s \in W$  such that  $x \in \text{fix}_\Theta(H, s)$ . By the invariance of  $\text{fix}_\Theta(H, s)$ , we get that

$$\omega(x) \subset \text{fix}_\Theta(H, s) \cap \text{fix}_\Theta(H, w),$$

showing that  $x \in \text{fix}_\Theta(H, w)$ . The proof for the unstable set is completely analogous.  $\square$

We note that  $\text{st}(\text{fix}_\Theta(H, 1))$  and  $\text{un}(\text{fix}_\Theta(H, w^-))$  are open and dense (see Section 2.2) so that  $\text{fix}_\Theta(H, 1)$  and  $\text{fix}_\Theta(H, w^-)$  are, respectively, the only attractor and repeller which are thus denoted by  $\text{fix}_\Theta^+(H)$  and  $\text{fix}_\Theta^-(H)$ . Using the previous result, we obtain the desired characterization of the finest Morse decomposition.

**Theorem 4.2** *Let  $g^t$  be a flow on  $\mathbb{F}_\Theta$  and  $g^t = e^t h^t u^t$  its Jordan decomposition. Each  $\text{fix}_\Theta(H, w)$  is chain transitive, so that  $\{\text{fix}_\Theta(H, w) : w \in W_H \setminus W/W_\Theta\}$  is the finest Morse decomposition. In particular, the chain recurrent set of  $g^t$  in  $\mathbb{F}_\Theta$  is given by*

$$\mathcal{R}_C(g^t) = \text{fix}_\Theta(h^t) = \bigcup_{w \in W} \text{fix}_\Theta(H, w).$$

**Proof:** By the connectedness of  $\text{fix}_\Theta(H, w)$  we only need to prove that each  $\text{fix}_\Theta(H, w)$  is chain recurrent. Let  $g^t = e^t u^t h^t$  be the Jordan decomposition of  $g^t$ . Note that the restriction of  $g^t$  to  $\text{fix}_\Theta(H, w)$  is given by  $e^t u^t$ . First we show that for each  $x \in \mathbb{F}_\Theta$  there exists  $y \in \mathbb{F}_\Theta$  such that  $u^t x \rightarrow y$ , when  $t \rightarrow \pm\infty$ . In fact,  $i(u^t x) = \rho(u)^t i(x)$ . By Lemma 3.2,  $\rho(u)$  is unipotent, so that, by Lemma A.6, there exists  $[v]$  such that  $i(u^t x) \rightarrow [v]$ , when  $t \rightarrow \pm\infty$ . Using that  $i$  is an embedding, there exists  $y \in \mathbb{F}_\Theta$  such that  $i(y) = [v]$ , which proves the claim. Now by Lemma 3.2,  $\rho(e^t)$  is elliptic, so it lies in a subgroup conjugated to  $O(V)$ . This allows us to choose a metric in  $V$  such that  $\rho(e^t)$  is an isometry for all  $t \in \mathbb{T}$ . This metric induces a metric in  $\mathbb{P}V$  and thus in  $\mathbb{F}_\Theta$ , by using the Plücker embedding, so that  $e^t$  is an isometry in  $\mathbb{F}_\Theta$ . By Lemma 2.2 applied to  $u^t, e^t$  it follows that  $g^t$  is chain recurrent on  $\text{fix}_\Theta(H, w)$ .  $\square$

We remark that  $H$  gives the parabolic type of  $g^t$  and  $Z_H$  gives the block reduction as defined in [14].

Now we obtain the desired characterization of the recurrent set.

**Theorem 4.3** *Let  $g^t$  be a flow on  $\mathbb{F}_\Theta$  and  $g^t = e^t h^t u^t$  its Jordan decomposition. Then the recurrent set of  $g^t$  in  $\mathbb{F}_\Theta$  is given by*

$$\mathcal{R}(g^t) = \text{fix}_\Theta(h^t) \cap \text{fix}_\Theta(u^t).$$



**Proof:** By Lemma 3.2 and by Theorem A.8, we have that

$$\mathcal{R}(\rho(g)^t) = \text{fix}(\rho(h)^t) \cap \text{fix}(\rho(u)^t).$$

Thus the result follows by noting that

$$i(\mathcal{R}(g^t)) = \mathcal{R}(\rho(g)^t) \cap i(\mathbb{F}_\Theta),$$

$$i(\text{fix}_\Theta(h^t)) = \text{fix}(\rho(h)^t) \cap i(\mathbb{F}_\Theta) \quad \text{and} \quad i(\text{fix}_\Theta(u^t)) = \text{fix}(\rho(u)^t) \cap i(\mathbb{F}_\Theta).$$

□

By using the previous characterization of the recurrent set, the following result computes the topological entropy of linearly induced flows on flag manifolds (see [16] for definition and properties of topological entropy).

**Theorem 4.4** *If  $g^t$  is a flow in  $\mathbb{F}_\Theta$ , with  $t \in \mathbb{Z}$ , then its topological entropy vanishes.*

**Proof:** By using the variational principle and Poincaré recurrence theorem (see [16]), the topological entropy of  $g^t$  coincides with the topological entropy of its restriction to the closure of its recurrent set. By Theorem 4.3, we have that the recurrent set of  $g^t$  is closed. Now let  $g^t = e^t h^t u^t$  be the Jordan decomposition of  $g^t$ . Using again Theorem 4.3, it follows that the restrictions of  $g^t$  and  $e^t$  to  $\mathcal{R}(g^t)$  coincide. Arguing exactly as in the proof of Theorem 4.2, we can provide a metric in  $\mathbb{F}_\Theta$  such that  $e^t$  is an isometry in  $\mathbb{F}_\Theta$ , for every  $t \in \mathbb{Z}$ . Thus the restriction of  $e^t$  to  $\mathcal{R}(g^t)$  is also an isometry and therefore its topological entropy vanishes. □

## 4.2 Conley index and structural stability

In this section, we first compute the Conley index of a linearly induced flow  $g^t$  on the flag manifold  $\mathbb{F}_\Theta$ . Then we characterize the linearly induced flows which are structurally stable.

We say that  $g^t$  is a conformal flow if  $u^t = 1$ , for all  $t \in \mathbb{T}$ , where  $g^t = e^t h^t u^t$  is its Jordan decomposition. For each Iwasawa decomposition  $G = KAN$  such that  $H \in \text{cl } \mathfrak{a}^+$ , we define the conformal subgroup given by the direct product  $C_H = K_H A_{\Theta(H)}$ . We say that the flow  $g^t$  has a conformal reduction if there exists a conformal subgroup such that  $g^t \in C_H$ , for all  $t \in \mathbb{T}$  (this is the kind of linear flow considered in [7]).

**Proposition 4.5** *The flow  $g^t$  is conformal if and only if it has a conformal reduction.*

**Proof:** First we work with  $\mathbb{T} = \mathbb{R}$ . Assume that  $g^t$  has a conformal reduction. Thus there exists a choice of an Iwasawa decomposition  $G = KAN$  with  $H \in \text{cl } \mathfrak{a}^+$  such that  $g^t = \exp(tX) \in C_H$ , for all  $t \in \mathbb{T}$ . Deriving at  $t = 0$  we get that  $X$  belongs to the Lie algebra of  $C_H$  which is given by the direct sum  $\mathfrak{k}_H \oplus \mathfrak{a}_{\Theta(H)}$ . Thus we can decompose  $X = \widehat{E} + \widehat{H}$  where  $\widehat{E} \in \mathfrak{k}_H$  commutes with  $\widehat{H} \in \mathfrak{a}_{\Theta(H)}$ . Since  $\text{ad}(\widehat{E})$  is  $B_\theta$ -anti-symmetric and  $\text{ad}(\widehat{H})$  is  $B_\theta$ -symmetric, it follows that  $X = \widehat{E} + \widehat{H}$  is the Jordan decomposition so that the nilpotent part  $N$  of  $X$  vanishes and thus  $u^t = 1$ , for all  $t \in \mathbb{T}$ . Conversely, assuming that  $u^t = \exp(tN) = 1$ , for all  $t \in \mathbb{T}$ , we have that  $N = 0$ , so that  $X = E + H$  is its Jordan decomposition. By Lemma 3.1, it follows that there exists an Iwasawa decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}^+$  with  $H \in \text{cl } \mathfrak{a}^+$  such that  $E \in \mathfrak{k}_H$  and  $H \in \text{cl } \mathfrak{a}^+$ . Hence  $g^t = \exp(tX) \in C_H$ , for all  $t \in \mathbb{T}$ .

Now we work with  $\mathbb{T} = \mathbb{Z}$ . Suppose that  $g^t$  has a conformal reduction. Hence there exists a choice of an Iwasawa decomposition  $G = KAN$  with  $H \in \text{cl } \mathfrak{a}^+$  such that  $g \in C_H$ . Thus we can decompose  $g = \widehat{e}\widehat{h}$  where  $\widehat{e} \in K_H$  commutes with  $\widehat{h} \in A_{\Theta(H)}$ . Since  $\widehat{e}$  is a  $B_\theta$ -isometry and  $\widehat{h}$  is  $B_\theta$ -positive, this is the multiplicative Jordan decomposition of  $g$  so that  $u^t = 1$ , for all  $t \in \mathbb{T}$ . Conversely, assume that  $u^t = 1$  so that  $g = eh$  is its multiplicative Jordan decomposition. By Lemma 3.1, it follows that there exists an Iwasawa decomposition  $G = KAN$  with  $H \in \text{cl } \mathfrak{a}^+$  such that  $e \in K_H$  and  $H \in \text{cl } \mathfrak{a}^+$ . Therefore  $g^t = \exp(tX) \in C_H$ , for all  $t \in \mathbb{T}$ .  $\square$

Given an Iwasawa decomposition of  $\mathfrak{g}$  we recall that we can embed the flag manifold  $\mathbb{F}_\Theta$  in  $\mathfrak{s}$  in the following way. Take  $H_\Theta \in \text{cl } \mathfrak{a}^+$  such that  $\Theta = \Theta(H_\Theta)$  and put

$$i : \mathbb{F}_\Theta \rightarrow \mathfrak{s}, \quad g\mathfrak{p}_\Theta \mapsto kH_\Theta,$$

where  $g \in G$  and  $g = kan$  is its Iwasawa decomposition, with  $k \in K$ ,  $a \in A$ ,  $n \in N$  (see Proposition 2.1 of [5]).

**Proposition 4.6** *Let the flow  $g^t$  be conformal. Then there exists an Iwasawa decomposition such that the height function of  $H$  with respect to the above embedding given by*

$$f : \mathbb{F}_\Theta \rightarrow \mathbb{R}, \quad x \mapsto B_\theta(i(x), H),$$

*is a Lyapunov function for the finest Morse decomposition.*

**Proof:** Since  $g^t$  is conformal, by the previous result there exists, an Iwasawa decomposition such that  $g^t \in C_H$ . Decompose  $g^t = e^t h^t$  with  $e^t \in K_H$  and  $h^t \in A_{\Theta(H)}$ . Note that height function  $l$  is  $K_H$ -invariant since for  $k \in K_H$ , we have

$$f(kx) = B_\theta(ki(x), H) = B_\theta(i(x), k^{-1}H) = B_\theta(i(x), H) = f(x),$$

where we used that  $k$  is an isometry with respect to  $B_\theta$ . Thus we have that

$$f(g^t x) = f(h^t x),$$

for all  $t \in \mathbb{T}$ , and the result follows by Proposition 3.3 item (ii) of [5].  $\square$

The next result gives further information about the finest Morse decomposition (see Theorem 3.2, Proposition 3.5 and Proposition 7.1 of [14]). For the definition of the flag manifold  $\mathbb{F}(\Delta)_{H_0}$  see Section 3.3 of [14].

**Proposition 4.7** *Let  $g^t$  be a flow on  $\mathbb{F}_\Theta$ . Put  $\Delta = \Theta(H)$  and take  $H_\Theta \in \text{cl}\mathfrak{a}^+$  such that  $\Theta = \Theta(H_\Theta)$ . Then  $\text{fix}_\Theta(H, w)$  is diffeomorphic to the flag manifold  $\mathbb{F}(\Delta)_{H_0}$ , where  $H_0$  is the orthogonal projection of  $wH_\Theta$  in  $\mathfrak{a}(\Delta)$ . Furthermore, the stable and unstable sets of  $\text{fix}_\Theta(H, w)$  are diffeomorphic to vector bundles over  $\mathbb{F}(\Delta)_{H_0}$ . Moreover, if  $g^t$  is a conformal flow, then it is normally hyperbolic and its restriction to  $\text{st}_\Theta(H, w)$  and  $\text{un}_\Theta(H, w)$  are conjugated to linear flows.*

In the next result we obtain the Conley index of the attractor and, when  $g^t$  is conformal, of all Morse components (see Proposition 7.6 and Corollary 7.8 of [14]).

**Theorem 4.8** *If  $\Theta(H) \subset \Theta$  or  $\Theta \subset \Theta(H)$ , then the Conley index of the attractor  $\text{fix}_\Theta^+(H)$  is the homotopy class of the flag manifold  $\mathbb{F}(\Delta)_{H_0}$ .*

*Let  $g^t$  be a conformal flow on  $\mathbb{F}_\Theta$  and consider the same notation of the previous proposition. Then the Conley index of the Morse component  $\text{fix}_\Theta(H, w)$  is the homotopy class of the Thom space of the vector bundle  $\text{un}_\Theta(H, w) \rightarrow \mathbb{F}(\Delta)_{H_0}$ . In particular, we have the following isomorphism in cohomology*

$$CH^{*+n_w}(\text{fix}_\Theta(H, w)) \simeq H^*(\mathbb{F}(\Delta)_{H_0}),$$

*where  $n_w$  is the dimension of  $\text{un}_\Theta(H, w)$  as a vector bundle. The cohomology coefficients are taken in  $\mathbb{Z}_2$  in the general case and in  $\mathbb{Z}$  when  $\text{un}_\Theta(H, w)$  is orientable.*

An element  $H \in \mathfrak{a}$  is said to be regular if there is no root  $\alpha \in \Sigma$  such that  $\alpha(H) = 0$ . An element  $X \in \mathfrak{g}$  is h-regular if  $H$  is regular in  $\mathfrak{a}$ , for some Iwasawa decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}^+$ , where  $X = E + H + N$  is its Jordan decomposition. Note that if  $X$  is h-regular then so is  $\psi X$  for  $\psi \in \text{Int}(\mathfrak{g})$ . An element  $g \in G$  is h-regular if  $H$  is regular in  $\mathfrak{a}$ , for some Iwasawa decomposition  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}^+$ , where  $g = ehu$  is its Jordan decomposition and  $\log h = \text{ad}(H)$ . Note that if  $g$  is h-regular, then so is  $hgh^{-1}$  for  $h \in G$ .

**Proposition 4.9** *The h-regular elements of  $\mathfrak{g}$  are dense in  $\mathfrak{g}$ , and the h-regular elements of  $G$  are dense in  $G$ .*

**Proof:** Fix a Cartan involution  $\theta$ . Let  $\mathfrak{j}$  be a  $\theta$ -stable Cartan subalgebra. Take a Cartan decomposition  $\mathfrak{j} = (\mathfrak{k} \cap \mathfrak{j}) \oplus (\mathfrak{s} \cap \mathfrak{j})$ . Since  $\mathfrak{j}$  is maximal abelian in  $\mathfrak{g}$ , it follows that  $\mathfrak{a}_{\mathfrak{j}} = \mathfrak{s} \cap \mathfrak{j}$  is maximal abelian in  $\mathfrak{s}$ . If  $X \in \mathfrak{j}$ , then its Cartan and Jordan decompositions coincide. In fact,  $\mathfrak{j}$  is abelian, an element in  $\mathfrak{k}$  is  $B_\theta$ -anti-symmetric and an element in  $\mathfrak{s}$  is  $B_\theta$ -symmetric. Now since the regular elements of  $\mathfrak{a}_{\mathfrak{j}}$  are dense in  $\mathfrak{a}_{\mathfrak{j}}$ , it follows that the h-regular elements of  $\mathfrak{j}$  are dense in  $\mathfrak{j}$ . By Proposition 1.3.4.1 p.101 of [17] there exist  $\theta$ -stable Cartan subalgebras  $\mathfrak{j}_1, \dots, \mathfrak{j}_r$  such that the set

$$\bigcup_{i=1}^r \text{Int}(\mathfrak{g})\mathfrak{j}_i$$

is dense in  $\mathfrak{g}$ . Thus it follows that the h-regular elements of  $\mathfrak{g}$  are dense in  $\mathfrak{g}$ .

Let  $J$  be a  $\theta$ -stable Cartan subgroup. By Proposition 1.4.1.2 p.109 of [17], we can take the Cartan decomposition

$$J = (J \cap K)(\exp(\mathfrak{j} \cap \mathfrak{s})).$$

Since  $\mathfrak{j}$  is maximal abelian in  $\mathfrak{g}$ , it follows that  $\mathfrak{a}_{\mathfrak{j}} = \mathfrak{s} \cap \mathfrak{j}$  is maximal abelian in  $\mathfrak{s}$ . If  $g \in J$ , then its Cartan and Jordan decompositions coincide. In fact,  $J$  centralizes  $\exp(\mathfrak{s})$ , an element in  $K$  is a  $B_\theta$ -isometry and an element in  $\exp(\mathfrak{s})$  is  $B_\theta$ -positive. Now since the regular elements of  $\mathfrak{a}_{\mathfrak{j}}$  are dense in  $\mathfrak{a}_{\mathfrak{j}}$ , it follows that the h-regular elements of  $J$  are dense in  $J$ . By Theorem 1.4.1.7 p.113 of [17] there exist  $\theta$ -stable Cartan subgroups  $J_1, \dots, J_r$  such that the set

$$\bigcup_{i=1}^r \bigcup_{g \in G} gJ_i g^{-1}$$

is dense in  $G$ . Thus it follows that the  $h$ -regular elements of  $G$  are dense in  $G$ .  $\square$

Note that a regular element is automatically conformal since we have that  $Z_H = MA$  when  $H$  is regular, where  $M$  is the centralizer of  $\mathfrak{a}$  in  $K$ . It follows that the above result implies Theorem 8.1 of [7].

**Lemma 4.10** *Let  $H \in \text{cl } \mathfrak{a}^+$  and  $\Theta \subset \Sigma$ . If  $H$  is not regular and  $\Theta \neq \Sigma$  then there exists  $w \in W$  such that  $\text{fix}_\Theta(H, w)$  is not an isolated point in  $\mathbb{F}_\Theta$ .*

**Proof:** Given  $w \in W$ , by Proposition 3.6 p.326 of [5] the map  $\varphi_w : w\mathfrak{n}_\Theta^- \rightarrow N_\Theta^- b_\Theta \subset \mathbb{F}_\Theta$ ,  $Y \mapsto \exp(Y)wb_\Theta$  is a diffeomorphism such that  $\varphi_w(0) = wb_\Theta$  and

$$\tilde{H} \circ \varphi_w = d\varphi_w(\text{ad}(H)|_{w\mathfrak{n}_\Theta^-}),$$

where  $\tilde{H}(x)$  denotes the induced field of  $H$  at  $x$ . Since  $\text{fix}_\Theta(H, w)$  consists of the connected set of zeroes of the induced vector field  $\tilde{H}$  which pass through  $wb_\Theta$ , it follows that

$$\varphi_w^{-1}(\text{fix}(H, w)_\Theta \cap N_\Theta^- b_\Theta) = \text{Ker } \text{ad}(H)|_{w\mathfrak{n}_\Theta^-}.$$

Now if  $H$  is not regular then there exists  $\alpha \in \Sigma$  such that  $\alpha(H) = 0$ . We have that

$$w\mathfrak{n}_\Theta^- = \sum \{\mathfrak{g}_{w^*\beta} : \beta \in \Pi^- - \langle \Theta \rangle\}.$$

Since  $\Theta \neq \Sigma$  we can take  $\beta \in \Pi^- - \langle \Theta \rangle$ . Since the Weyl group  $W$  acts transitively on  $\Pi$  we can take  $w \in W$  such that  $w^*\beta = \alpha$ . Since  $\alpha(H) = 0$  it follows that  $\mathfrak{g}_\alpha \subset \text{Ker } \text{ad}(H)|_{w\mathfrak{n}_\Theta^-}$ . From the above discussion it follows that

$$\varphi_w(\mathfrak{g}_\alpha) \subset \text{fix}_\Theta(H, w),$$

so that  $\text{fix}_\Theta(H, w)$  is not an isolated point in  $\mathbb{F}_\Theta$ .  $\square$

Now we obtain the desired characterization of the linearly induced flows which are structurally stable. For the concepts of structural stability and of Morse-Smale flows and diffeomorphisms see [11]. A flow  $g^t$  in  $\mathbb{F}_\Theta$  is regular if  $H$  is regular in  $\mathfrak{a}$ , where  $g^t = e^t h^t u^t$  is its Jordan decomposition.

**Theorem 4.11** *Let  $g^t$  be a flow on  $\mathbb{F}_\Theta$ , where  $\Theta \subset \Sigma$  with  $\Theta \neq \Sigma$ . Then the following conditions are equivalent:*

- (i)  $g^t$  is regular,
- (ii)  $g^t$  is Morse-Smale and
- (iii)  $g^t$  is structurally stable.

**Proof:** First we show that condition (i) implies condition (ii). Let  $g^t = e^t h^t u^t$  be the Jordan decomposition of  $g^t$  and  $H = \log h$ . If  $H$  is regular, then for each  $w \in W$ , the set  $\text{fix}_\Theta(H, w)$  reduces to a point. Thus, by Theorem 4.2, it follows that  $\mathcal{R}_C(g^t)$  is a finite set of fixed points. By Proposition 4.7, we have that each point in  $\mathcal{R}_C(g^t)$  is hyperbolic. Furthermore, by Lemma 4.2 of [5] p.331, the stable and unstable manifolds, given by Proposition 4.1, intersect transversally. Thus it follows that  $g^t$  is Morse-Smale. By a result of [11], it follows that condition (ii) implies condition (iii).

Now we show that the negation of condition (i) implies the negation of condition (iii). If  $H$  is not regular, for each  $\Theta \subset \Sigma$  with  $\Theta \neq \Sigma$ , there exists a Morse component  $\text{fix}_\Theta(H, w)$  which has infinite points. Thus, by Theorem 4.2, the same is true for  $\mathcal{R}_C(g^t)$ . When  $\mathbb{T} = \mathbb{Z}$ , let  $\hat{g} \in G$  be a  $h$ -regular element arbitrarily close to  $g$ , given by Proposition 4.9, so that  $\hat{g}^t$  is a regular flow. Then  $\hat{g}$  is arbitrarily close to  $g$  in  $\text{Diff}(\mathbb{F}_\Theta)$ . By the first part of the proof, we have that  $\mathcal{R}_C(\hat{g}^t)$  is finite and thus  $\hat{g}^t$  and  $g^t$  cannot be topologically equivalent. This proves that, in this case,  $g^t$  is not structurally stable. In the case  $\mathbb{T} = \mathbb{R}$ , we have that  $g^t = \exp(tX)$ , for some  $X \in \mathfrak{g}$ . Let  $\hat{X} \in \mathfrak{g}$  be an  $h$ -regular element arbitrarily close to  $X$ , given by Proposition 4.9, so that  $\hat{g}^t = \exp(t\hat{X})$  is a regular flow. Then  $\hat{X}$  is arbitrarily close to  $X$  in  $\mathcal{X}(\mathbb{F}_\Theta)$ . Using again the first part of the proof, it follows that  $\mathcal{R}_C(\hat{g}^t)$  is finite and thus  $\hat{g}^t$  and  $g^t$  cannot be topologically equivalent. Thus we have again that  $g^t$  is not structurally stable.  $\square$

**Remark:** Here we show how the results on the flag manifolds recover the analogous ones on the projective space. Consider the Lie algebra  $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$  and the Lie group  $G = \text{Int}(\mathfrak{g}) = \text{Ad}(\text{Sl}(n, \mathbb{R}))$ . A canonical choice of Iwasawa decomposition gives the maximal abelian  $\mathfrak{a} = \text{diagonal matrices with trace zero}$  and roots  $\Pi$  given by the functionals in  $\mathfrak{a}$  defined by  $\alpha_{ij}(\text{diag}(H_k)) = H_i - H_j$ ,  $i \neq j$ ,  $i, j = 1, \dots, n$ . Fixing the set of simple roots  $\Theta = \{\alpha_{i, i+1} : i = 2, \dots, n-1\}$ , then the corresponding parabolic subgroup in  $\text{Sl}(n, \mathbb{R})$  is

$$P_\Theta = \begin{pmatrix} a & v \\ 0 & A \end{pmatrix}, \quad a \in \mathbb{R}, \quad v \in \mathbb{R}^{n-1} \text{ and } \text{tr} A + a = 0,$$

which is precisely the isotropy subgroup at  $[e_1]$  of the canonical action of  $\mathrm{Sl}(n, \mathbb{R})$  in  $\mathbb{P}\mathbb{R}^n$ . It follows that the map

$$\varphi : \mathbb{P}\mathbb{R}^n \rightarrow \mathbb{F}_\Theta, \quad T[e_1] \mapsto \mathrm{Ad}(T)\mathfrak{p}_\Theta, \quad T \in \mathrm{Sl}(n, \mathbb{R}),$$

is an equivariant diffeomorphism. For  $X \in \mathfrak{g}$ , the map  $\varphi$  conjugates the action of  $e^{tX}$  on  $\mathbb{P}\mathbb{R}^n$  with the action of  $\exp(tX)$  on  $\mathbb{F}_\Theta$ , since

$$\varphi(e^{tX}T[e_1]) = e^{t\mathrm{ad}(X)}\varphi(T[e_1]) = \exp(tX)\varphi(T[e_1]).$$

We claim that  $X \in \mathfrak{sl}(n, \mathbb{R})$  is h-regular if the real part of its eigenvalues are distinct. In fact, writing the Jordan decomposition  $X = E + H + N$ ,  $H \in \mathrm{cl} \mathfrak{a}^+$ , then the entries of  $H = \mathrm{diag}(H_1, \dots, H_n)$  are the real part of the eigenvalues of  $X$ . The claim follows, since  $H$  is regular in  $\mathfrak{a}$  when  $\alpha_{ij}(H) = H_i - H_j \neq 0$ .

## 5 Floquet theory

In the previous sections, from the point of view of differential equations, we have considered equations with constant coefficients. In this section we extend these results to equations with periodic coefficients. Throughout this section we fix  $\mathbb{T} = \mathbb{R}$ .

The fundamental solution associated with a given continuous map  $t \in \mathbb{T} \mapsto X(t) \in \mathfrak{g}$  is the map  $t \in \mathbb{T} \mapsto g(t) \in G$  which satisfies

$$g'(t) = X(t)g(t)$$

and  $g(0) = I$ . It is straightforward to show that

$$\rho_s : \mathbb{T} \rightarrow G, \quad t \mapsto \rho_s(t) = g(t+s)g(s)^{-1}$$

is the fundamental solution associated with the map  $t \in \mathbb{T} \mapsto X(s+t) \in \mathfrak{g}$ . When  $X$  is constant, we have that  $g(t) = g^t = \exp(tX)$ , which is the flow induced on  $G$  by the right-invariant vector field  $X^r(a) = Xa$ , where  $a \in G$ .

When  $X$  is  $T$ -periodic  $g(t)$  is not in general a flow. Despite of this, we can associate a flow to  $g(t)$  in the following way. Since  $X(t+T) = X(t)$ , we have that  $\rho_T(t) = g(t)$ , for all  $t \in \mathbb{T}$ . Thus it follows that  $g(t+T) = g(t)g(T)$  and that  $g(t+mT) = g(t)g(T)^m$ . We will need the following result which can be regarded as a refinement of Floquet's lemma to semisimple Lie groups (see Theorem 2.47, p.163 of [3]).

**Lemma 5.1** *Let  $G = \text{Int}(\mathfrak{g})$ . For  $g \in G$  there exist  $m \in \mathbb{N}$  and  $X \in \mathfrak{g}$  such that  $g^m = \exp(X)$ .*

**Proof:** Let  $g = ehu$  be its Jordan decomposition. By Lemma 3.1 there exists an Iwasawa decomposition  $G = KAN$  such that  $h = e^{\text{ad}(H)}$ ,  $H \in \text{cl } \mathfrak{a}^+$  and  $e \in K_H$ . We have that  $u = I + T$  where  $T$  is a nilpotent map. Since  $e, h$  commute with  $u$ , it follows that  $e, h$  commute with  $T$ . By Lemma IX.7.3 p.431 of [6], we have that  $u = e^{\text{ad}(N)}$  where  $N \in \mathfrak{g}$  is such that  $\text{ad}(N)$  is nilpotent. By Lemma VI.4.5 p.270 of [6],  $\text{ad}(N) = \log(u) = \log(I + T)$  which is a polynomial in  $T$ . It follows that  $e, h$  commute with  $\text{ad}(N)$  and thus  $e \in K_N$ . We claim that  $H$  commutes with  $N$ . Note that if  $p(x)$  is a polynomial, then  $p(h)$  commutes with  $\text{ad}(N)$ . There exists a basis such that  $\text{ad}(H)$  and  $h = e^{\text{ad}(H)}$  are given respectively by the diagonal matrices  $\text{diag}(\lambda_1, \dots, \lambda_n)$  and  $\text{diag}(e^{\lambda_1}, \dots, e^{\lambda_n})$ . There exists  $a, b > 0$  such that interval  $[a, b]$  contains the eigenvalues of  $h$ . By the Weistrass approximation theorem, there exists a sequence of polynomials  $p_n(x)$  such that for  $x \in [a, b]$  we have  $p_n(x) \rightarrow \log(x)$ . Thus we have that  $p_n(h) \rightarrow \text{ad}(H)$ , which shows that  $\text{ad}(H)$  commutes with  $\text{ad}(N)$ , since  $p_n(h)$  commutes with  $\text{ad}(N)$  for all  $n \in \mathbb{N}$ .

From the above considerations it follows that  $e$  lies in the compact group  $L = K_H \cap K_N$  which has Lie algebra  $\mathfrak{k}_H \cap \mathfrak{k}_N$ . It follows that there exists  $m \in \mathbb{N}$  such that  $e^m$  in the connected component  $L$  containing the identity. Thus, by Lemma II.6.10 p.135 of [6], there exist  $E \in \mathfrak{k}_H \cap \mathfrak{k}_N$  such that  $e^m = \exp(E)$ . Taking  $X = E + mH + mN$  it follows that

$$\exp(X) = \exp(E) \exp(H)^m \exp(N)^m = e^m h^m u^m = g^m.$$

□

By the above result, there exists  $X \in \mathfrak{g}$  such that  $g(T)^m = \exp(mTX)$ . Defining  $g^t = \exp(tX)$  and

$$a(t) = g(t)g^{-t},$$

it is straightforward to check that  $a(t + mT) = a(t)$  and that

$$\rho_s(t) = a(t + s)g^t a(s)^{-1},$$

for all  $t \in \mathbb{T}$ . We have that the map defined by

$$\phi^t(s, a) = (s + t, \rho_s(t)a)$$



is a flow of automorphisms on the principal bundle  $S^1 \times G$ . In fact, observing that the map  $(s, a) \mapsto (s, a(s)a)$  is a diffeomorphism of  $S^1 \times G$  onto itself, it follows that

$$\phi^t(s, a(s)a) = (s + t, a(s + t)g^t a)$$

and thus is not hard to check that  $\phi^t$  is already a flow.

From now on we consider the Jordan decomposition  $g^t = e^t h^t u^t$  of the flow on  $\mathbb{F}_\Theta$  associated to  $\phi^t$ , where  $\log h = \text{ad}(H)$  and  $H$  lies in the closure of a fixed Weyl chamber  $\mathfrak{a}^+$  (see Section 3). For each  $\mathbb{F}_\Theta$ , we can induce a flow on  $S^1 \times \mathbb{F}_\Theta$  by simply putting

$$\phi^t(s, x) = (s + t, \rho_s(t)x)$$

or

$$\phi^t(s, a(s)x) = (s + t, a(s + t)g^t x).$$

Note that when  $X(t)$  is constant and equals to  $X$ , then  $a(t) = I$  and  $\phi^t(s, x) = (s + t, g^t x)$ , so we return to the same context of the previous section.

Now we obtain the desired characterization of the recurrent set.

**Theorem 5.2** *The recurrent set of  $\phi^t$  in  $S^1 \times \mathbb{F}_\Theta$  is given by*

$$\mathcal{R}(\phi^t) = \{(s, a(s)x) : s \in S^1, x \in \text{fix}_\Theta(h^t) \cap \text{fix}_\Theta(u^t)\}.$$

**Proof:** Denoting by

$$R = \{(s, a(s)x) : s \in S^1, x \in \text{fix}_\Theta(h^t) \cap \text{fix}_\Theta(u^t)\},$$

we will first show that  $R \subset \mathcal{R}(\phi^t)$ . Given  $(s, a(s)x)$ , where  $x \in \mathcal{R}(g^t)$ , by Theorem 4.3, we have that  $g^t x = e^t x$ , for all  $t \in \mathbb{T}$ . Arguing exactly as in the proof of Theorem 4.2, we can provide a metric  $d$  in  $\mathbb{F}_\Theta$  such that  $e^t$  is an isometry in  $\mathbb{F}_\Theta$ , for every  $t \in \mathbb{T}$ . By the compactness of  $\mathbb{F}_\Theta$  and by Lemma 2.1, there exists a sequence  $n_k \rightarrow \infty$  such that  $g^{n_k m T} x \rightarrow x$ . It follows that

$$\phi^{n_k m T}(s, a(s)x) = (n_k m T + s, a(n_k m T + s)g^{n_k m T} x) = (s, a(s)g^{n_k m T} x) \rightarrow (s, a(s)x),$$

showing that  $R \subset \mathcal{R}(\phi^t)$ . Conversely, let  $(s, a(s)x) \in \mathcal{R}(\phi^t)$ . Thus there exists  $t_k \rightarrow \infty$  such that

$$(t_k + s, a(t_k + s)g^{t_k} x) = \phi^{t_k}(s, a(s)x) \rightarrow (s, a(s)x).$$

Therefore  $t_k + s \rightarrow s$  modulo  $mT$  so that  $a(t_k + s) \rightarrow a(s)$  and thus  $g^{t_k}x \rightarrow x$ .  $\square$

Let  $\{\text{fix}_\Theta(H, w) : w \in W_H \backslash W/W_\Theta\}$  be the finest Morse decomposition of the flow  $g^t$  given by Theorem 4.2 and define

$$\mathcal{M}_\Theta(H, w) = \{(s, a(s)x) : s \in S^1, x \in \text{fix}_\Theta(H, w)\},$$

which is a  $\phi^t$ -invariant subset of  $S^1 \times \mathbb{F}_\Theta$ . If  $f : \mathbb{F}_\Theta \rightarrow \mathbb{R}$  is a Lyapunov function for the finest Morse decomposition of  $g^t$  and defining

$$F(s, a(s)x) = f(x),$$

we have that  $F : S^1 \times \mathbb{F}_\Theta \rightarrow \mathbb{R}$  is a Lyapunov function for the family

$$\{\mathcal{M}_\Theta(H, w) : w \in W_H \backslash W/W_\Theta\},$$

which is, therefore, a Morse decomposition of the flow  $\phi^t$ . In fact,

$$F(\phi^t(s, a(s)x)) = F(s + t, a(s + t)g^tx) = f(g^tx)$$

and thus  $F \circ \phi^t$  is constant over each  $\mathcal{M}_\Theta(H, w)$  and strictly decreasing out of their union. Now we characterize the stable and unstable sets of these Morse components.

**Proposition 5.3** *The stable and unstable sets of  $\mathcal{M}_\Theta(H, w)$  are given by*

$$\text{st}(\mathcal{M}_\Theta(H, w)) = \{(s, a(s)x) : s \in S^1, x \in \text{st}_\Theta(H, w)\}$$

and

$$\text{un}(\mathcal{M}_\Theta(H, w)) = \{(s, a(s)x) : s \in S^1, x \in \text{un}_\Theta(H, w)\}.$$

**Proof:** Taking  $x \in \text{st}_\Theta(H, w)$ , by Proposition 4.1, we have that  $g^tx \rightarrow \text{fix}_\Theta(H, w)$ . Thus it follows that

$$\phi^t(s, a(s)x) = (s + t, a(s + t)g^tx) \rightarrow \text{st}(\mathcal{M}_\Theta(H, w)),$$

showing that  $(s, a(s)x) \in \text{st}(\mathcal{M}_\Theta(H, w))$ . The equality follows by observing that, by the Bruhat decomposition, the sets  $\text{st}_\Theta(H, w)$ ,  $w \in W$ , exhaust  $\mathbb{F}_\Theta$ . The proof for the unstable set is entirely analogous.  $\square$

We denote the only attractor and the only repeller  $\mathcal{M}_\Theta(H, 1)$  and  $\mathcal{M}_\Theta(H, w^-)$ , respectively, by  $\mathcal{M}_\Theta^+(H)$  and  $\mathcal{M}_\Theta^-(H)$ . Using the previous result, we obtain the desired characterization of the finest Morse decomposition.

**Theorem 5.4** *Each  $\mathcal{M}_\Theta(H, w)$  is chain transitive, so that  $\{\mathcal{M}_\Theta(H, w) : w \in W_H \setminus W/W_\Theta\}$  is the finest Morse decomposition of  $\phi^t$ . In particular, the chain recurrent set of  $\phi^t$  in  $S^1 \times \mathbb{F}_\Theta$  is given by*

$$\mathcal{R}_C(\phi^t) = \{(s, a(s)x) : s \in S^1, x \in \text{fix}_\Theta(h^t)\}.$$

**Proof:** First assume that, for all given  $\varepsilon > 0$ ,  $t \in \mathbb{T}$ , and  $x, y \in \text{fix}_\Theta(H, w)$ , there is an  $(\varepsilon, t)$ -chain from  $(0, x)$  to  $(s, a(s)y)$ . With this we will construct an  $(\varepsilon, t)$ -chain from  $(\widehat{s}, a(\widehat{s})x)$  to  $(\widetilde{s}, a(\widetilde{s})y)$ , for all  $x, y \in \text{fix}_\Theta(H, w)$ . In fact, take  $\delta > 0$  given by the  $\varepsilon$ -uniform continuity of  $\phi^{\widehat{s}}$  in  $S^1 \times \mathbb{F}_\Theta$ . Denote by  $\overline{d}$  the metric in  $S^1 \times \mathbb{F}_\Theta$  given by

$$\overline{d}((s, x), (r, y)) = |s - r| + d(x, y),$$

where  $d$  is a metric in  $\mathbb{F}_\Theta$ . Consider the  $(\delta, t)$ -chain from  $(0, x)$  to

$$(\widetilde{s} - \widehat{s}, a(\widetilde{s} - \widehat{s})g^{-\widehat{s}}y),$$

given by  $t_i > t$  and  $\eta_i \in S^1 \times \mathbb{F}_\Theta$ , where  $i = 1, \dots, n+1$  such that  $\overline{d}(\eta_i, \phi^{t_i}(\eta_i)) < \delta$ , for  $i = 1, \dots, n$ . We have thus that, with the same  $t_i > t$ ,  $\phi^{\widehat{s}}(\eta_i)$  is an  $(\varepsilon, t)$ -chain from  $(\widehat{s}, a(\widehat{s})x)$  to  $(\widetilde{s}, a(\widetilde{s})y)$ .

Now we will prove the above assumption. Let  $x, y \in \text{fix}_\Theta(H, w)$  and take  $\delta > 0$  given by the  $\varepsilon$ -uniform continuity of the map  $(s, z) \mapsto a(s)z$  in  $S^1 \times \mathbb{F}_\Theta$ . By the compactness of  $\mathbb{F}_\Theta$ , there exists  $\tau > 0$  such that  $d(g^t z, z) < \delta/2$ , for all  $t \in [0, \tau]$  and all  $z \in \mathbb{F}_\Theta$ . By Theorem 4.2, there exists a  $(\delta/2, t)$ -chain from  $x$  to  $y$  in  $\mathbb{F}_\Theta$  given by  $t_i > t$  and  $x_i \in \mathbb{F}_\Theta$ , where  $i = 1, \dots, n+1$ . Note that  $n$  can be taken arbitrarily large such that  $mT/n < \tau$ . Let  $l$  be such that

$$\widehat{s} = s + lmT - (t_1 + \dots + t_n) \in [0, mT].$$

Consider  $\widehat{t} = \widehat{s}/n \in [0, \tau]$  and define  $\widehat{t}_i = t_i + \widehat{t} > t$ ,

$$\xi_1 = (0, x) \quad \text{and} \quad \xi_{i+1} = (\widehat{t}_1 + \dots + \widehat{t}_i, a(\widehat{t}_1 + \dots + \widehat{t}_i)x_{i+1}).$$

We claim that this provides an  $(\varepsilon, t)$ -chain from  $(0, x)$  to  $(s, a(s)y)$ . In fact, note that

$$\widehat{t}_1 + \dots + \widehat{t}_n = s + lmT,$$

and that  $x_{n+1} = y$ , so that we have  $\xi_{n+1} = (s, a(s)y)$ . Since

$$\phi^{\widehat{t}_i}(\xi_i) = (\widehat{t}_1 + \dots + \widehat{t}_i, a(\widehat{t}_1 + \dots + \widehat{t}_i)g^{\widehat{t}_i}x_i)$$

and

$$d\left(g^{\widehat{t}_i}x_i, x_{i+1}\right) \leq d\left(g^{\widehat{t}}z, z\right) + d\left(g^{t_i}x_i, x_{i+1}\right) < \delta,$$

where  $z = g^{t_i}x_i$ , we have that

$$\overline{d}\left(\phi^{\widehat{t}_i}(\xi_i), \xi_{i+1}\right) < \varepsilon.$$

□

Consider the set

$$Q_\phi = \{(s, a(s)a) : s \in S^1, a \in Z_H\} \subset S^1 \times G.$$

Since  $g^t \in Z_H$ , it follows that  $Q_\phi$  is a  $\phi^t$ -invariant  $Z_H$ -reduction of the principal bundle  $S^1 \times G$  which in [14] has been called a block reduction of the flow  $\phi^t$ . Looking at  $S^1 \times \mathbb{F}_\Theta$  as an associated bundle of  $S^1 \times G$ , by the previous results, this reduction has the following immediate properties

$$\mathcal{M}_\Theta(H, w) = Q_\phi \cdot wb_\Theta \quad \text{and} \quad \text{st}(\mathcal{M}_\Theta(H, w)) = Q_\phi \cdot \text{st}_\Theta(H, w),$$

recovering, in this context, Theorem 4.1 item (i) and Theorem 5.3 of [14].

We say that the flow  $\phi^t$  is conformal if its associated flow  $g^t$  is conformal. In this case, by Proposition 4.5, there exists a conformal subgroup  $C_H$  of  $Z_H$  which contains  $g^t$ . It follows that the set

$$C_\phi = \{(s, a(s)c) : s \in S^1, c \in C_H\} \subset S^1 \times G$$

is a  $\phi^t$ -invariant  $C_H$ -reduction of the principal bundle  $S^1 \times G$  which in [14] has been called a conformal reduction of the flow  $\phi^t$ . Note that both  $Q_\phi$  and  $C_\phi$  have a global section given by  $s \mapsto (s, a(s))$ . Thus, by Corollary 8.4, p.36 of [15], these are trivial bundles, implying that their associated bundles are also trivial. The next result gives further information about the finest Morse decomposition (see Propositions 4.2, 6.2 and 7.1 of [14]). For the definition of the flag manifold  $\mathbb{F}(\Delta)_{H_0}$  see Section 3.3 of [14].

**Proposition 5.5** *Put  $\Delta = \Theta(H)$  and take  $H_\Theta \in \text{cl}\mathfrak{a}^+$  such that  $\Theta = \Theta(H_\Theta)$ . Then  $\mathcal{M}_\Theta(H, w)$  is homeomorphic to  $S^1 \times \mathbb{F}(\Delta)_{H_0}$ , where  $H_0$  is the orthogonal projection of  $wH_\Theta$  in  $\mathfrak{a}(\Delta)$ . Furthermore, the stable and unstable sets of  $\text{fix}_\Theta(H, w)$  are diffeomorphic to vector bundles over  $\mathcal{M}_\Theta(H, w)$ . Moreover, if  $\phi^t$  is a conformal flow, then it is normally hyperbolic and its restriction to  $\text{st}(\mathcal{M}_\Theta(H, w))$  and  $\text{un}(\mathcal{M}_\Theta(H, w))$  are conjugated to linear flows.*

In the next result we obtain the Conley index of the attractor and, when  $\phi^t$  is conformal, of all Morse components (see Theorem 7.4 and Corollary 7.8 of [14]).

**Theorem 5.6** *If  $\Theta(H) \subset \Theta$  or  $\Theta \subset \Theta(H)$ , then the Conley index of the attractor  $\mathcal{M}_\Theta^+(H)$  is its homotopy class.*

*If  $\phi^t$  is conformal, then the Conley index of the Morse component  $\mathcal{M}_\Theta(H, w)$  is the homotopy class of the Thom space of the vector bundle  $\text{un}(\mathcal{M}_\Theta(H, w)) \rightarrow \mathcal{M}_\Theta(H, w)$ . In particular, we have the following isomorphism in cohomology*

$$CH^{*+n_w}(\mathcal{M}_\Theta(H, w)) \simeq H^*(S^1 \times \mathbb{F}(\Delta)_{H_0}),$$

*where  $n_w$  is the dimension of  $\text{un}(\mathcal{M}_\Theta(H, w))$  as a vector bundle. The cohomology coefficients are taken in  $\mathbb{Z}_2$  in the general case and in  $\mathbb{Z}$  when  $\text{un}_\Theta(H, w)$  is orientable.*

It follows that the (co)homology Conley indexes can be computed by K uneth formula, once we know the (co)homology of the real flag manifolds. For the homology of real flag manifolds see [9].

## A Dynamics in projective spaces

In this appendix, we relate the Jordan decomposition of  $g^t$  in  $\text{Gl}(V)$  to the dynamics of the induced linear flow  $g^t$  on the projective space  $\mathbb{P}V$ , where  $V$  is a finite dimensional vector space. The main results of the section deals with the characterization of the recurrent set and the finest Morse decomposition in terms of the fixed points of the Jordan components. We will need some preliminary lemmas.

**Lemma A.1** *Let  $V = U \oplus W$  and let  $x_n = u_n + w_n$  be a sequence with  $u_n \in U$  and  $w_n \in W$ . Suppose that  $u_n \neq 0$  for all  $n \in \mathbb{N}$  and that  $\lim \frac{w_n}{|u_n|} = 0$ . Passing to the projective space, if  $[x_n] \in \mathbb{P}V$  converges to  $[x] \in \mathbb{P}V$  then  $[x] \in \mathbb{P}U$ .*

**Proof:** Without loss of generality we can suppose that  $|u_n| = 1$  and that  $\lim w_n = 0$  since  $[x_n] = [\frac{u_n}{|u_n|} + \frac{w_n}{|u_n|}]$  and  $\lim \frac{w_n}{|u_n|} = 0$ . Now since  $|u_n| = 1$  it has a convergent subsequence  $u_{n_k} \rightarrow u$ , where  $u \in U$ , since subspaces are closed in  $V$ . Then

$$[x] = \lim_{k \rightarrow \infty} [x_{n_k}] = \lim_{k \rightarrow \infty} [u_{n_k} + w_{n_k}] = [u]$$

and thus  $[x] \in \mathbb{P}U$ . □

**Lemma A.2** *Fix a norm  $|\cdot|$  in  $V$ . If  $h = I$  then for each  $x \neq 0$  there exists  $\epsilon > 0$  such that  $|g^t x| > \epsilon$  for all  $t \in \mathbb{T}$ .*

**Proof:** By the Jordan canonical form, in an appropriate basis,  $u$  is upper triangular with ones on the diagonal. Write  $x$  in this basis as  $x = (x_1, \dots, x_k, 0, \dots, 0)$  where  $x_k$  is the last nonzero coordinate of  $x$ . Then  $u^t$  fixes the last coordinate  $x_k$  of  $x$  so that, if we take the euclidian norm  $|\cdot|_1$  relative to this basis, we have that  $|u^t x|_1 \geq |x_k|$  for all  $t \in \mathbb{T}$ . Taking the norm  $|\cdot|_2$  which makes  $e$  an isometry, we have that  $|v|_2 \geq C|v|_1$  for all  $v \in V$ , where  $C > 0$ . Since by hypothesis  $g^t = e^t u^t$ , we have that

$$|g^t x|_2 = |u^t x|_2 \geq C|u^t x|_1 \geq C|x_k|$$

for all  $t \in \mathbb{T}$ . Using that  $|\cdot|_2$  is equivalent to the norm  $|\cdot|$  fixed in  $V$ , the lemma follows. □

For the following lemma, we need to recall the definition of the spectral radius of a linear map  $g$ , which is given by

$$r(g) = \max\{|\lambda| : \lambda \text{ is an eigenvalue of } g\}.$$

Let  $V$  have a norm  $|\cdot|$ . We also denote by  $|g|$  the corresponding operator norm.

**Lemma A.3** *If  $r(g) < 1$  then  $|g^t| \rightarrow 0$ .*

**Proof:** Let  $g^t = e^t h^t u^t$  be the Jordan decomposition of the linear flow  $g^t$ . Since the norms in  $V$  are equivalent, we can choose  $|\cdot|$  such that the eigenvector basis of  $h$  is orthonormal. Then we have that  $|h^t| = r(g)^t$ . Since  $e$  is elliptic, it lies in a subgroup conjugated to  $O(V)$  so that the norm  $|e^t|$  is bounded, say by  $M > 0$ . Since  $u$  is unipotent, we have that  $u = e^N$ , where  $N$  is nilpotent, so that

$$u^t = e^{tN} = I + tN + \dots + (tN)^k/k!,$$

for some fixed  $k \in \mathbb{N}$ . Using that  $|N^l| \leq |N|^l$ , it follows that for  $v \in V$

$$|u^t v| \leq |v|(1 + |N|t + \dots + (|N|^k/k!)t^k) = |v|p(t),$$

where  $p(t)$  is a polynomial in  $t$ , so that  $|u^t| \leq p(t)$ . Collecting the above results, we have that

$$|g^t| \leq |e^t| |h^t| |u^t| \leq Mr(g)^t p(t) \rightarrow 0,$$

when  $t \rightarrow \infty$ , since  $r(g) < 1$ . □

**Lemma A.4** *Let  $V = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_n}$  be the eigenspace decomposition of  $V$  associated to  $h$  where  $\lambda_1 > \lambda_2 > \cdots > \lambda_n > 0$ . Let  $v = v_1 + v_2 + \cdots + v_n$ ,  $v \neq 0$ , with  $v_i \in V_{\lambda_i}$ . Take  $i$  the first index  $k$  with  $v_k \neq 0$  and  $j$  the last index  $k$  with  $v_k \neq 0$ . Then  $\omega([v]) \subset \mathbb{P}V_i$  and  $\omega^*([v]) \subset \mathbb{P}V_j$ .*

**Proof:** Denote by  $g_k$  the restriction of  $g/\lambda_i$  to  $V_{\lambda_k}$ . We have that  $g_k$  has spectral radius  $\lambda_k/\lambda_i$  and that  $g_i$  has hyperbolic part equal to the identity. By Lemma A.3 we have that for  $k > i$  we have  $|g_k^t v_k| \leq |g_k^t| |v_k| \rightarrow 0$ , when  $t \rightarrow \infty$ . By Lemma A.2 we have that  $|g_i v_i| \geq \epsilon$ , for some  $\epsilon > 0$ . Now let  $t_j \rightarrow \infty$  be such that  $\lim_{j \rightarrow \infty} g^{t_j}[v] = [x]$ , then

$$g^{t_j}[v] = \left[ \frac{g^{t_j}}{\lambda_i} (v_i + \cdots + v_n) \right] = \left[ g_i^{t_j} v_i + \sum_{k>i} g_k^{t_j} v_k \right] \rightarrow [x],$$

so that, by the previous arguments and Lemma A.1, it follows that  $[x] \in \mathbb{P}V_i$ . □

With the previous result we obtain that the projective eigenspaces of the hyperbolic part of  $g^t$  give are Morse components for the flow  $g^t$  in the projective space.

**Proposition A.5** *The set  $\{\mathbb{P}V_{\lambda_1}, \dots, \mathbb{P}V_{\lambda_n}\}$  is a Morse decomposition. Furthermore, the stable set of  $\mathbb{P}V_{\lambda_i}$  is given by*

$$\text{st}(\mathbb{P}V_{\lambda_i}) = \{[v_i + v_{i+1} + \cdots + v_n] : v_i \neq 0\},$$

where  $v_k \in V_{\lambda_k}$ .

**Proof:** Since  $h^t$  commutes with  $g^t$  and taking  $v_k \in V_{\lambda_k}$ , it follows that

$$h^t g^t v_k = g^t h^t v_k = \lambda_k g^t v_k,$$

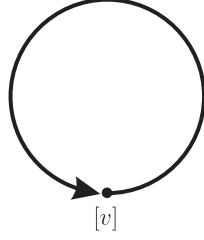


Figure 1: Unipotent element acting on the projective line.

showing that  $V_{\lambda_k}$  is  $g^t$ -invariant. The proposition then follows from the definition of Morse decomposition, using the previous lemma.  $\square$

In order to show that the above Morse decomposition is the finest one, we need to consider the behavior of the unipotent component of  $g^t$ . This is done in the next lemma, which generalizes the behavior on the projective line (see Figure 1) of the action of the linearly induced map  $e^{tN}$ , where

$$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

**Lemma A.6** *Let  $x \neq 0$  be a vector and  $N$  be a nilpotent transformation. If  $k$  is such that  $N^{k+1}x = 0$  and  $v = N^k x \neq 0$ , then  $e^{tN}[x] \rightarrow [v]$ , when  $t \rightarrow \pm\infty$ , where  $t \in \mathbb{T}$ . Furthermore  $e^{tN}v = v$ , for all  $t \in \mathbb{T}$ .*

**Proof:** First note that

$$e^{tN}v = \left( \sum_{j \geq 0} \frac{t^j}{j!} N^j \right) N^k x = v + \sum_{j \geq 1} \frac{t^j}{j!} N^{k+j} x = v.$$

Now we have that

$$e^{tN}[x] = \left[ x + tN + \cdots + \frac{t^k}{k!} v \right]$$

and, multiplying by  $k!/t^k$ , we get that

$$e^{tN}[x] = \left[ v + \frac{k!}{t^k} \left( tNx + \cdots + \frac{t^{k-1}}{(k-1)!} N^{k-1}x \right) \right] \rightarrow [v],$$



when  $t \rightarrow \pm\infty$ , where  $t \in \mathbb{T}$ . □

Collecting the previous results we obtain the desired characterization of the finest Morse decomposition.

**Theorem A.7** *Let  $g : V \rightarrow V$  be a linear isomorphism, where  $V$  is a finite dimensional vector space. Let  $V = V_{\lambda_1} \oplus V_{\lambda_2} \oplus \cdots \oplus V_{\lambda_n}$  be the eigenspace decomposition of  $V$  associated to  $h$ . Then each  $\mathbb{P}V_{\lambda_i}$  is chain transitive, so that  $\{\mathbb{P}V_{\lambda_1}, \dots, \mathbb{P}V_{\lambda_n}\}$  is the finest Morse decomposition. In particular, the chain recurrent set of  $g$  in  $\mathbb{P}V$  is given by*

$$\mathcal{R}_C(g^t) = \text{fix}(h^t) = \bigcup_i \mathbb{P}V_{\lambda_i}.$$

**Proof:** By the connectedness of  $\mathbb{P}V_{\lambda_i}$  we just need to prove that each  $\mathbb{P}V_{\lambda_i}$  is chain recurrent. We note that the second equality on the equation above is immediate. Thus, by Proposition A.5 we have that  $\mathcal{R}_C(g) \subset \text{fix}(h^t)$ . Now we prove that  $\text{fix}(h^t)$  is chain recurrent. First note that the restriction of  $g^t$  to  $\text{fix}(h^t)$  is given by  $e^t u^t$ . Since  $e^t$  is elliptic, it lies in a subgroup conjugated to  $O(V)$ . This allows us to choose a metric in  $V$  such that  $e^t$  is an isometry for all  $t \in \mathbb{T}$ . This metric induces a metric in  $\mathbb{P}V$  such that  $e^t$  is an isometry in  $\mathbb{P}V$ . By Lemmas A.6 and 2.2 applied to  $u^t$ ,  $e^t$  it follows that  $g^t$  is chain recurrent on  $\text{fix}(h^t)$ . □

We conclude with the desired characterization of the recurrent set.

**Theorem A.8** *Let  $g : V \rightarrow V$  be a linear isomorphism, where  $V$  is a finite dimensional vector space. Then the recurrent set of  $g$  in  $\mathbb{P}V$  is given by*

$$\mathcal{R}(g^t) = \text{fix}(h^t) \cap \text{fix}(u^t).$$

**Proof:** Let  $[x]$  be such that  $[x] \in \omega([x])$ . By Theorem A.7 we have that  $[x] \in \text{fix}(h^t)$ . Let  $t_j \rightarrow \infty$  be such that  $g^{t_j}[x] \rightarrow [x]$ . Since  $e^t$  is elliptic, it lies in a subgroup conjugated to  $O(V)$ , so we can assume that  $e^{t_j} \rightarrow E$ . Note that  $E$  commutes with the Jordan components of  $g$ . By Lemma A.6, there exists a fixed point  $[v]$  of  $u^t$  such that  $u^{t_j}[x] \rightarrow [v]$ . Since  $g^{t_j} = e^{t_j} u^{t_j} h^{t_j}$  it follows that

$$[x] = \lim g^{t_j}[x] = \lim e^{t_j} u^{t_j}[x] = E[v].$$

The theorem follows since  $E$  commutes with  $u^t$  and  $[v]$  is a  $u^t$ -fixed point.  $\square$

We illustrate the above results with some examples in dimension three. In order to stay in the context of semisimple Lie groups we work in  $\mathrm{Sl}(3, \mathbb{R})$ . **Example:** Let  $X \in \mathfrak{sl}(3, \mathbb{R})$ . There exists  $g \in \mathrm{Sl}(3, \mathbb{R})$  such that  $gXg^{-1}$  has one of the following Jordan canonical forms, where  $a, b \in \mathbb{R}$ :

$$X_1 = \begin{pmatrix} -a & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & a+b \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X_4 = \begin{pmatrix} -a & -b & 0 \\ b & -a & 0 \\ 0 & 0 & 2a \end{pmatrix} \quad \text{and} \quad X_5 = \begin{pmatrix} -a & 1 & 0 \\ 0 & -a & 0 \\ 0 & 0 & 2a \end{pmatrix}.$$

Let  $a, b > 0$ . We have that the nilpotent component of  $X_4$  is zero, while its elliptic and hyperbolic components are given, respectively, by

$$E = \begin{pmatrix} 0 & -b & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad H = \begin{pmatrix} -a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & 2a \end{pmatrix}.$$

In this case, the recurrent and chain recurrent sets coincide and we have two Morse components: the attractor  $[e_3]$  and the repeller  $[\mathbb{R}e_1 \oplus \mathbb{R}e_2]$  (see Figure 2(a)). We also have that the elliptic component of  $X_5$  is zero, while its hyperbolic and nilpotent components are given, respectively, by

$$H = \begin{pmatrix} -a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & 2a \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In this case, the recurrent and chain recurrent sets are different, but the Morse components remain the same. The recurrent set is given by  $\{[e_1], [e_3]\}$  (see Figure 2(b)). In both cases, the stable set of the attractor is the complement of the repeller.

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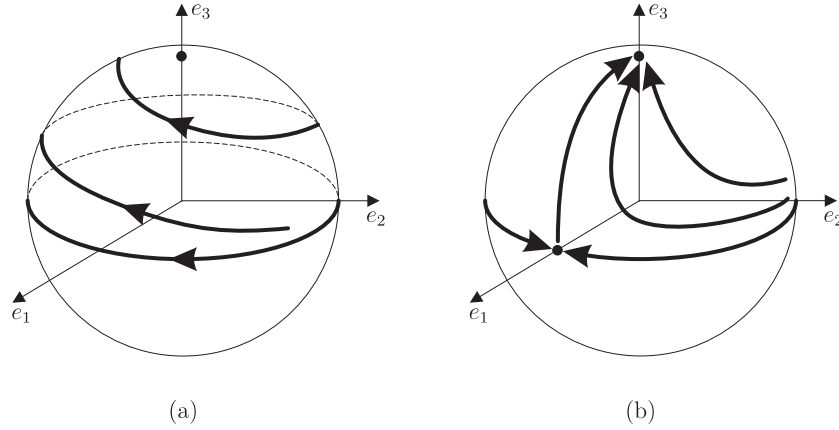


Figure 2: Dynamics on the projective space represented on the two-sphere.

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